Stockholm University

Gravity Beyond a Single Metric

Ghost-free interactions of spin-2 fields

Joakim Flinckman

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Abstract

The mathematical formulations of the fundamental interactions in nature are all expressed in terms of fields classified by their mass and spin. The structure of two of the most successful theories in physics, General Relativity and the Standard Model of particle physics, can both be derived more or less uniquely from fields with fixed mass and spin, guided by theoretical consistency conditions. The Standard Model contains well-understood interactions of particles with spins less than two, while theories involving spins higher than two are generally problematic. Theories of spin-2 fields are much less explored, and in particular, any consistent multiplet structure, crucial for the Standard Model, is practically unknown.

Even without considering multiplet structures, classical theories with interacting spin-2 fields generically propagate additional pathological fields, so-called Boulware–Deser ghosts, rendering such theories physically unacceptable. Until recently, only a non-derivative interaction between two spin-2 fields, called bimetric theory, had been proven consistent. While the bimetric interaction can to some extent be trivially extended to more than two fields, genuine ghost-free multi-field interactions have remained elusive.

In this thesis, we derive necessary conditions on ghost-free non-derivative interactions between multiple spin-2 fields. With a suitable parametrisation of the fields, the ghost mode can be isolated, allowing one to derive necessary conditions for it to be non-propagating. These conditions essentially determine the form of the interaction potential uniquely, up to a set of free constant parameters. The admissible class contains all previously known ghost-free theories, but is not itself ghost-free in general. Imposing an additional necessary condition for ghost-freedom on these parameters further restricts the interaction to the Hassan–Schmidt–May theory and trivial extensions thereof. Thus, much as for the lower-spin theories in the Standard Model, the requirement of theoretical consistency, here the absence of ghosts, essentially singles out the admissible theory. We then show that this theory possesses the full set of constraints needed to eliminate all ghost modes, establishing it as the most general known ghost-free multi-gravity interaction. Finally, we derive the mass spectrum, finding one massless and multiple massive spin-2 fields with a non-tachyonic spectrum.

Keywords: *multi-gravity, modified gravity, bimetric theory, spin-2 fields, classical field theory.*

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Sammanfattning

Denna avhandling utforskar teorier som innefattar växelverkan mellan spin-2-fält. Spin-2-fält är tätt sammankopplade med gravitationskraften och Einsteins allmänna relativitetsteori. Utöver att revolutionera vår förståelse av rum och tid har den allmänna relativitetsteorin under det senaste århundradet varit extremt framgångsrik i att förutspå och beskriva vitt skilda fenomen såsom svarta hål, gravitationsvågor samt hur universum beter sig på de allra största skalorna. Men det finns fortfarande många öppna frågor som rör gravitationen. Till exempel finns det experimentellt stöd för fenomen såsom mörk energi och mörk materia, som båda är tätt kopplade till gravitationen, men vars ursprung ännu är okänt.

Einsteins teori är inte heller förenlig med den andra grundpelaren inom modern fysik, kvantmekaniken. De andra fundamentala naturkrafterna, den starka och den elektrosvaga växelverkan, beskrivs av uppsättningar av flera kvantfält som interagerar och genom symmetritransformationer blandas med varandra. Dessa typer av strukturer är essentiella, både ur ett teoretiskt och experimentellt perspektiv. Einsteins teori saknar dessa strukturer och innehåller enbart ett enda fält. Byggstenarna för den starka och den elektrosvaga växelverkan kan i princip härledas unikt enbart från fysikaliska och matematiska argument. Likaså kan matematiska argument användas för att härleda Einsteins allmänna relativitetsteori, där gravitationen beskrivs som ett masslöst spin-2-fält.

I denna avhandling beskrivs hur teorier med flera spin-2-fält kan formuleras. Den matematiska formen av sådana teorier är starkt begränsad av matematiska och teoretisk-fysikaliska principer, i synnerhet kravet på att inga patologiska, så kallade Boulware–Deser-spöken, får förekomma. En teori som innehåller dessa spöken bör förkastas och anses inte vara fysikalisk. Men det har visat sig svårt att formulera dessa typer av teorier utan spöken. Med hjälp av en lämplig parametrisering av fälten kan spökfälten dock isoleras. Detta gör det möjligt att härleda nödvändiga villkor för den spökfria växelverkans form. Dessa villkor begränsar mängden teorier som potentiellt kan vara spökfria till en unik form, och med en vidare analys kan denna mängd begränsas till en specifik teori, Hassan–Schmidt–May-teorin samt triviala utvidgningar av den. I likhet med teorierna i standardmodellen leder alltså kravet på teoretisk konsistens i stort sett till en unik teori. Vi visar sedan att denna teori har rätt antal frihetsgrader och att spökfälten elimineras fullständigt. Slutligen härleds fältens masspektrum, vilket visar att teorin verkligen beskriver ett masslöst och flera massiva spin-2-fält med positiva massor.

List of Papers

Paper I J. Flinckman, S. F. Hassan, *On the Uniqueness of Ghost-Free Multi-Gravity – II: Constraining antisymmetrised multi spin-2 interactions*, Preprint, [arXiv:2604.07625]

Paper II J. Flinckman, D. Blixt, *Canonical Vielbeins for General Relativity: $D + 1$ Decomposition and Constraint Analysis*, Preprint, [arXiv:2602.18491]

Paper III J. Flinckman, S. F. Hassan, *Existence of ghost-eliminating constraints in multivielbein theory*, Preprint, [arXiv:2510.03014]

Paper IV J. Flinckman, S. F. Hassan, *Mass spectrum & linear perturbations of ghost-free multi-spin-2 theory*, JHEP 02 (2025) 176, [arXiv:2410.09439]

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Preface

This dissertation is organised in two parts. Part I is an overview that sets out the necessary background and places the main results of the included works in context. Part II contains the associated papers in full. A brief chapter-by-chapter outline of Part I is provided below.

Outline

Chapter 1 provides an introduction to the subject and its motivations, together with a brief historical overview of theories of spin-2 fields. The chapter concludes with notation and conventions used throughout the thesis.

Chapter 2 introduces constrained systems, first in the Lagrangian framework and then in the Hamiltonian formulation via the Dirac–Bergmann algorithm, and concludes with the classification into first- and second-class constraints, constraint algebra, symmetry generators and the dimensionality of the physical phase space.

Chapter 3 reviews Einstein's theory of General Relativity, first in a covariant formulation and then in a canonical form. Both the standard metric variables and an SO(3)-covariant canonical vielbein formulation are presented, including the Hamiltonian field equations, constraint structure and symmetry generators. In particular, the boost generator is constructed, thereby restoring full local Lorentz symmetry in the canonical vielbein formulation.

Chapter 4 presents the unimodular decomposition of the $3 + 1$ metric variables, which isolates the mode associated with the Boulware–Deser ghost in non-linear interacting spin-2 theories. Using this decomposition, the ghost is isolated through a simplifying Ansatz, which allows one to derive necessary conditions on non-derivative interactions between spin-2 fields with Einstein–Hilbert-type kinetic terms. These conditions are then used to derive the most general locally Lorentz invariant interaction potential.

Chapter 5 derives necessary conditions on the interaction parameters of the admissible potential for the theory to be ghost-free. It is shown that the constraint structure generically does not allow for the ghost to be eliminated, and the conditions under which this obstruction is avoided are derived. For irreducible interactions, these conditions single out the Hassan–Schmidt–May determinant coupling as the unique consistent multi-gravity interaction, while for two fields, Bimetric Theory is recovered. The chapter concludes with a discussion of reducible interactions and the uniqueness of ghost-free non-derivative multi-gravity theories.

Chapter 6 presents the Hassan–Schmidt–May theory in its covariant and canonical forms. Using the canonical vielbein formalism, the total Hamiltonian is constructed and the Dirac–Bergmann algorithm is carried out, including the identification of the ghost-eliminating constraints and the symmetry generators. After classifying the constraints, the dimensionality of the physical phase space is computed, showing that the pathological ghost-modes are absent.

Chapter 7 introduces the perturbative treatment of General Relativity, its quadratic action, and its relation to a massless spin-2 field, and then extends the discussion to massive spin-2 fields. The Lorentz-covariant perturbative expansion of a vielbein in terms of metric perturbations and Lorentz fields is reviewed and applied to the multi-gravity potential. The mass spectrum and mass eigenstates of the Hassan–Schmidt–May theory are presented and shown to be non-tachyonic, yielding one massless and $\mathcal{N}-1$ healthy massive spin-2 fields and no scalar ghost-modes, in agreement with the non-linear counting of propagating degrees of freedom.

Chapter 8 concludes the thesis with a summary and outlook.

Contribution to papers

Paper I
The original idea emerged from discussions between the authors. I performed the computations, derived the necessary conditions and their implications for the results, and the manuscript was written in collaboration with S. F. Hassan.

Paper II
The work developed through regular discussions between the authors and intermittent meetings with S. F. Hassan. I carried out the development of the formalism and derivations and wrote the manuscript in the form included in this thesis.

Paper III
The main results arose from regular discussions between the authors. I performed the calculations, proved the existence of the additional constraints, and wrote the text, apart from Section 2.1.

Paper IV
I performed the calculations and wrote the majority of the text. In particular, I proved the existence of solutions to the background consistency conditions and derived the form of the eigenstates and the restrictions on the eigenvalues.

Material from licentiate thesis

Some parts of the abstract and introduction are reused from the licentiate thesis *Beyond Bimetric Theory: Consistent Theories of Multiple Spin-2 Fields*, 2023 [1] (unpublished) but are heavily modified and extended. No other sections are reused.

Use of Artificial Intelligence

The AI language models ChatGPT 5.4 and Claude Opus 4.6 have been used during the preparation of this thesis, exclusively for spell-checking, grammatical corrections, and minor linguistic improvements and reformulations. No part of the scientific content, including derivations, arguments, or conclusions was generated by these tools.

Contents

Part I

Chapter 1

Introduction

1.1 Background

It is a remarkable fact that mathematics can describe the fundamental laws of nature. As a result, theoretical physicists often study mathematical structures as a way to search for and develop new physical theories. In particular, examining these structures' mathematical and theoretical consistency has been a valuable method for identifying promising models and eliminating those that are not physically viable. An early example is James Clerk Maxwell's introduction of a displacement current in Ampère's circuit law, which reconciled Gauss's law and the conservation of electric charge [2]. Even though Maxwell's argument for adding the term came from mathematical consistency, it predicted physical phenomena which were later experimentally verified.

During the 20th century, the power of theoretical physics and the imposition of theoretical consistency yielded predictions and constructions such as Dirac's prediction of the positron [3], the formulation of Yang–Mills theory [4], the proposal of the quark model [5, 6], and the theoretical basis for the Higgs field [7–11], to name only a few. These developments were all driven by theoretical considerations, and some of them, such as Dirac's and Yang–Mills, were formulated without any prior experimental evidence. Yet, they later turned out to form central building blocks of the modern understanding of the fundamental laws of nature. This thesis focuses on the *theoretical consistency* of classical field theories of modified gravity called multi-gravity.

In 1916, Einstein's theory of General Relativity provided a novel perspective on gravitational physics and the structure of space and time [12]. Not only did General Relativity provide a geometric description of the universe, with the gravitational degrees of freedom encoded in a spacetime metric, but it proved extraordinarily fruitful for cosmology and predicted new physical phenomena such as gravitational waves, black holes and gravitational lensing, all subsequently confirmed observationally. Einstein formulated his theory without any direct empirical evidence, but by trying to conceptually reconcile his newly discovered Special Relativity with Newton's gravitational theory and to find a consistent mathematical framework. After the initial introduction and later empirical success of General Relativity, Lovelock showed that Einstein's theory is essentially the unique theory of a spacetime metric [13, 14], furthering the confidence in the theory.

Despite its success, many questions remain unanswered in gravitational physics. In particular, while General Relativity has been remarkably successful, it raises additional questions about the universe's constituents, where a large majority of the energy content appears to come from unknown sources called dark energy and dark matter. Yet perhaps the most puzzling question is how gravity fits together with the other fundamental laws of nature. Efforts to merge General Relativity with a quantised worldview have been a major endeavour since the advent of quantum mechanics and remain an open problem to this day.

Quantum field theory, particularly the Standard Model of particle physics, is another theory that has proved itself admirably successful and predictive. While General Relativity and the Standard Model are in some sense incompatible, they share some structural characteristics. They can both be formulated in terms of fields classified by their mass and spin, the so-called Wigner classification [15]. For fields of definite mass and spin, the form of free field equations can essentially be determined uniquely by imposing the absence of so-called ghosts, pathological fields with negative kinetic energy, which render a theory containing them unstable and physically unacceptable. This purely theoretical consistency condition can be used to derive the well-known theories of Klein–Gordon (spin-0), Dirac (spin-$1/2$), and Maxwell/Proca (spin-1), which form the fundamental building blocks of the Standard Model. For spin-2, the unique ghost-free theory was formulated by Fierz–Pauli in [16, 17] in 1939. The massless Fierz–Pauli theory corresponds precisely to General Relativity linearised around flat space. However, for spins larger than 2 there are fundamental obstacles, and beyond the quadratic level, local interacting theories of a higher-spin field are widely believed not to exist [18]. By contrast, interacting spin-1 fields are essentially fixed to be Yang–Mills theories, and consistent self-interactions of a massless spin-2 field are fixed to be General Relativity [19].

The Standard Model contains multiple fields of varying masses and spins 0, $1/2$, and 1, interacting with each other. Some fields with identical spin are arranged in so-called gauge multiplets, i.e. they are grouped together and transform into each other under so-called gauge transformations. For instance, leptons, like the electron and electron neutrino, are arranged into SU(2) doublets, while the Electroweak gauge bosons form a triplet and a singlet giving rise to the photon, $W^{\pm}$ and Z particles. This intricate construction is crucial for the theoretical and phenomenological consistency of the theory and mathematical consistency can strongly constrain, and in some cases essentially uniquely determine the set of viable theories [20–23]. These structures and their consistency were discovered well before they were phenomenologically justified, yet predicted many now-discovered physical features of the universe and the Standard Model.

In contrast, the structure of multiplets, so important in the Standard Model, is absent in General Relativity, where the spacetime metric is intricately linked to a single massless spin-2 field. This raises the prospect that gravity might be part of a more comprehensive theory encompassing a wider range of physical phenomena. In other words, is it possible that gravity is not a stand-alone force, but instead, only one aspect of a more complete framework, similar to how electromagnetism is part of the electroweak theory? Studying these types of structures could potentially shed light on some of the open questions related to gravity, such as dark matter, dark energy or even quantum gravity.

Past experience suggests that one could introduce additional spin-2 fields and possibly impose multiplet structures similar to those found in the Standard Model. Introducing multiple spin-2 fields also allows us to sidestep Lovelock's uniqueness of General Relativity, which only applies to one massless field, creating a more general class of gravitational theories. While the study of multiplet structures for spins lower than 2 is well-understood, theories of multiple spin-2 fields still need to be explored. The idea of theories with more than one metric or spin-2 field is nothing new. The seminal papers of Fierz and Pauli [16, 17] provided the quadratic action for a spin-2 field, which was later generalised by Aragone and Deser to couple it to gravity [24], thereby laying the foundation for further work in multi-gravity. An independent idea was proposed by Nathan Rosen in the early 1940s, where spacetime was endowed with one Lorentzian metric, like General Relativity, accompanied by an additional Euclidean metric [25]. Closer to the modern theories of multi-gravity are Isham, Salam and Strathdee's [26] attempt to write down consistent interactions between gravity and massive spin-2 mesons, Chamseddine, Salam and Strathdee's attempt to write down a gauge symmetry between gravitons [27], or Zumino's attempt to describe spin-2 hadrons interacting with gravity [28].[1] The latter three all contain massive spin-2 fields, but in light of Boulware and Deser's result [29, 30] that any non-linear theory with a massive spin-2 always propagates a ghostly scalar mode, interest in such theories diminished.

The additional pathological field, now called the Boulware–Deser ghost, appears when non-linear terms are added to Fierz and Pauli's theory and was long thought to be inevitable [31]. However, this conclusion was shown to be incorrect in 2010, when de Rham, Gabadadze and Tolley studied higher-order extensions to the Fierz–Pauli action and identified the loophole in the previous analysis [32, 33]. The resulting non-linear theory, now called Massive Gravity (or sometimes dRGT-gravity after de Rham, Gabadadze and Tolley), was later generalised and proved to be ghost-free to all orders by Hassan, Rosen and Schmidt–May

[1] Interestingly, one of the interaction terms that Zumino wrote down would later be shown to be a ghost-free interaction in contradiction to Boulware and Deser's results.

[34–36].[2] Since there are restrictions on consistent interactions between massless spin-2 fields [38, 39], the existence of a non-linear massive spin-2 field opens the possibility for an interacting theory with more than one spin-2 field.

A key feature of Massive Gravity is that the mass term for the metric $g_{\mu\nu}$ necessarily involves a second metric $f_{\mu\nu}$ as non-trivial contractions such as $g_{\mu\nu}f^{\mu\nu}$ require a reference structure upon which the massive field propagates. In the form in which Hassan and Rosen had written the Massive Gravity interaction, it was soon realised that $f_{\mu\nu}$ could be elevated to a dynamical field, leading to the creation of Bimetric Theory, a ghost-free theory of two interacting spin-2 fields [36, 40–42]. Research on both Massive Gravity and Bimetric Theory is ongoing, and reviews of the subjects are provided by [43, 44] and [45].

This thesis and the work preceding it go beyond Bimetric Theory and consider theories with more than two spin-2 fields and their theoretical consistency in terms of the absence of ghosts. In particular, we are concerned with $\mathcal{N}$ interacting metrics $g^I_{\mu\nu}$ and theories with an action of the form,

$$\mathcal{S} = \sum_{I=1}^{\mathcal{N}} \mathcal{S}_{\text{EH}}[g_I] + \mathcal{S}_{\text{int}}[g_1, \dots, g_{\mathcal{N}}], \tag{1.1}$$

where $\mathcal{S}_{\text{EH}}$ are standard Einstein–Hilbert terms, which individually yield $\mathcal{N}$ copies of General Relativity and the metrics interact with one another via $\mathcal{S}_{\text{int}}$. A wide class of theories can be generated even by considering only non-derivative interactions in $\mathcal{S}_{\text{int}}$, and we investigate the theoretical consistency of such multi-gravity theories. The central finding is that ghost-freedom restricts the allowed interactions, singling out a unique class of consistent theories from the much larger space of possibilities. This echoes the broader pattern in theoretical physics, where imposing theoretical consistency severely constrains the space of physically viable theories.

1.2 Notation and Conventions

We will work in units where $c = \hbar = 1$, and adopt the curvature sign convention of Wald [46]. Throughout the thesis, excluding Section 2, there will be up to five sets of sub- and superscripts. Greek letters, $\alpha, \beta, \dots$ denote *coordinate indices* of 3+1 dimensional spacetime and run from 0 to 3. Lowercase Latin letters from the middle of the alphabet, $i, j, k, \dots$ run from 1 to 3 and correspond to *spatial coordinate indices*. Letters from the beginning of the Latin alphabet will denote indices on the *internal Lorentz space* (Lorentz indices), where capital letters $A, B, C, \dots$ run from 0 to 3, and lower case letters $a, b, c, \dots$ run from 1 to 3 and denote the *spatial internal space*. All these indices are subject to the Einstein summation convention, in contrast to the *species labels*, denoted by capital characters from the middle of the Latin alphabet $I, J, K, \dots$, where repeated indices are *not* implicitly summed. These labels come as both super- or subscripts, but there is no covariant structure, i.e. $X_I = X^I$, and they will generically be placed where notationally convenient.

We adopt the *mostly plus signature* for Lorentzian metrics, so that the Minkowski metric is,

$$\eta_{AB} = \text{diag}(-1, +1, +1, +1), \tag{1.2}$$

and its spatial components $\eta_{ab} = \delta_{ab} = \text{diag}(+1, +1, +1)$ correspond to the flat Euclidean metric. Symmetrisation and antisymmetrisation brackets are normalised as,

$$X_{(\alpha\beta)} = \tfrac{1}{2}\left[X_{\alpha\beta} + X_{\beta\alpha}\right], \qquad Y_{[\alpha\beta]} = \frac{1}{2}\left[Y_{\alpha\beta} - Y_{\beta\alpha}\right]. \tag{1.3}$$

Lorentz indices are raised (lowered) by η^{AB} (η_{AB}) or δ^{ab} (δ_{ab}) if they are spatial. We will, however, be careful with raising and lowering coordinate indices in sections covering multi-gravity since,

$$g_I^{\mu\nu} X_\nu \neq g_J^{\mu\nu} X_\nu, \qquad g^I_{\mu\nu} Y^\nu \neq g^J_{\mu\nu} Y^\nu, \tag{1.4}$$

for different metrics g_I and g_J. We will make use of the 3- and 4-dimensional totally antisymmetric epsilon symbols $\epsilon_{0123} = \epsilon^{0123} = +1$ and $\epsilon_{123} = \epsilon^{123} = +1$, and the generalised Kronecker delta,

$$\delta^{\alpha\beta\gamma\delta}_{ABCD} = \epsilon_{ABCD}\epsilon^{\alpha\beta\gamma\delta}. \tag{1.5}$$

Chapter 2

Constrained Systems

This chapter reviews some aspects of constrained mechanics, first in terms of the Lagrangian formalism, then in the Hamiltonian, applying the Dirac–Bergmann algorithm [47–49]. We finish the chapter with the classification of constraints and the notion of a first-class algebra and symmetry generators.

2.1 Constrained Lagrangian Mechanics

Many physical systems and theories can be formulated in terms of an action,[1]

$$\mathcal{S}[q^i(t)] = \int \mathrm{d}t\, L(q^i(t), \dot{q}^i(t)), \tag{2.1}$$

where the equations of motion for the $\mathcal{N}$ generalised coordinates $q^i(t)$ are derived using the *principle of stationary action*, i.e. the physical trajectories $q^i(t)$ are those for which the action functional is stationary under infinitesimal variations $\delta q^i(t)$ of the trajectories. The trajectories can be derived by imposing that the variation of the action $\delta\mathcal{S}$ vanish to first order in $\delta q^i(t)$,

$$\delta\mathcal{S} = \mathcal{S}[q^i + \delta q^i] - \mathcal{S}[q^i] \tag{2.2}$$

$$= \int \mathrm{d}t\, F_j(q^i(t), \dot{q}^i(t), \ddot{q}^i(t))\delta q^j + \mathcal{O}(\delta q^2) = 0. \tag{2.3}$$

Since the variations $\delta q^i(t)$ are arbitrary but keep the endpoints fixed, $\delta\mathcal{S}$ vanishes only if the solutions obey $F_j = 0$. The equation $F_i = 0$ has a closed form in terms of the Lagrangian via the *Euler–Lagrange equations*, yielding a set of second-order differential equations,

$$F_i = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j = 0. \tag{2.4}$$

[1] The following formalism can easily be generalised to contain higher order time derivatives of q^i or to field theories where $q^i(t) \mapsto \psi_a(x)$.

If the matrix $M_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is invertible, all the equations are second order in time derivatives, and one can explicitly solve for the accelerations,

$$\ddot{q}^k = (M^{-1})^{ki} \left(\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j \right), \tag{2.5}$$

which in turn can be integrated given the initial conditions to yield the solutions $q^i(t)$.

However, if M_{ij} is not invertible, there exist linear combinations of the equations $F_i = 0$ which are at most first order in time derivatives. Such equations are non-dynamical and are *constraints* on the coordinates $q^i(t)$ and their velocities $\dot{q}^i(t)$. This can be seen explicitly from the fact that if M_{ij} has a non-trivial kernel, there exist functions $u^i_a(q, \dot{q})$, such that (2.4) implies,

$$0 = u^i_a M_{ij} \ddot{q}^j = u^i_a \left(\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j \right) = \mathcal{C}_a(q^i, \dot{q}^i), \tag{2.6}$$

where a enumerates the number of linearly independent vectors in the kernel of M_{ij}. The non-dynamical equations $\mathcal{C}_a = 0$ constrain the velocities $\dot{q}^i(t)$ and coordinates $q^i(t)$. It might happen that one or more $\mathcal{C}_a = 0$ can only be used to eliminate coordinates that have a second-order equation of motion of the form (2.5), yielding two equations for the same coordinate. Combining these, the previously dynamical equation (2.5) instead yields additional constraints.

In the case where $\mathcal{C}_a$ is trivially zero, not all coordinates can be determined by the equations of motion, and the system generically has so-called *gauge symmetries*. Gauge symmetries arise when linear combinations of the equations of motion and their time-derivatives vanish identically. In such cases, the undetermined coordinates remain arbitrary and different choices, or gauge choices, do not affect the physical content of the theory.

2.2 Constrained Hamiltonian Mechanics

It is often convenient to express a theory in terms of the Hamiltonian formulation, by defining conjugate momenta,

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.7}$$

Given that one can invert (2.7) for the velocities $\dot{q}^i$ in terms of q^i and p_i, the Hamiltonian can be constructed via the Legendre transform,

$$H(q^i, p_i) = p_i \dot{q}^i - L(q^i, \dot{q}^i(q^k, p_k)). \tag{2.8}$$

This is not possible if,

$$\delta p_i = \delta \left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \delta \dot{q}^j = M_{ij} \delta \dot{q}^j, \tag{2.9}$$

is not invertible and the system is again constrained.

For concreteness, we will consider a Lagrangian which is at most quadratic in $\dot{q}^i$, or can be put in that form by partial integration, e.g. terms like $q\ddot{q}$. The Lagrangian then takes the form,

$$L = \tfrac{1}{2}\dot{q}^i M_{ij}(q)\dot{q}^j - A_i(q)\dot{q}^i - V(q), \tag{2.10}$$

and the conjugate momenta,

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = M_{ij}\dot{q}^j - A_i. \tag{2.11}$$

If M_{ij} is singular, we cannot solve for all the velocities $\dot{q}^j$ uniquely and eliminate them in terms of the momenta p_i. We can, however, resort to the linear algebra notion of a *generalised inverse* M^+, which is defined by the property,

$$(M^+M)^i{}_j = P^i{}_j, \qquad\qquad M_{ik}P^k_j = M_{ij}, \tag{2.12}$$

where P is the projector onto the image of M.[2] The generalised inverse is not unique, but it is convenient to also impose the conditions,

$$M^+MM^+ = M^+, \qquad\qquad MM^+M = M, \tag{2.13a}$$
$$(MM^+)^{\mathrm{T}} = MM^+, \qquad\qquad (M^+M)^{\mathrm{T}} = M^+M, \tag{2.13b}$$

so that M^+ becomes the so-called Moore–Penrose *pseudoinverse*.

The singular system (2.11) has a particular solution for $\dot{q}^i$ in terms of p_i, obtained by contracting with the pseudoinverse,

$$\begin{aligned}(M^+)^{ij}p_j &= (M^+)^{ij}(M_{jk}\dot{q}^k - A_j), \qquad \Longrightarrow \\ P^i{}_j\dot{q}^j &= (M^+)^{ij}(p_j + A_j).\end{aligned} \tag{2.14}$$

By direct substitution of (2.14) into (2.11), this solves (2.11), but it is not the general solution. As M_{ij} is singular, there exist functions $u^i_a(q^j)$, which span the kernel of M, and fulfil,

$$M_{ij}u^j_a\lambda^a = 0, \tag{2.15}$$

for arbitrary functions λ^a. Any such combination can be added to (2.14) and it remains a solution. So the general solution for the velocities also contains contributions $\lambda^a u^i_a$,

$$\dot{q}^i = (M^+)^{ij}(p_j + A_j) + \lambda^a u^i_a, \tag{2.16}$$

[2] In general, P must be a projector onto the image of the adjoint $M^\dagger$, given some inner product, which corresponds to a set of covectors, but since M is symmetric, and we here assume a trivial covariant structure for $i, j, \ldots$, P^i_j is also the projector onto the image of M.

for arbitrary λ^a.

By direct substitution we can express the Lagrangian in terms of the momenta,

$$L = \tfrac{1}{2}p_i(M^+)^{ij}p_j - \tfrac{1}{2}A_i(M^+)^{ij}A_j - V - \lambda^a u_a^i A_i, \tag{2.17}$$

and the canonical one-form follows by multiplying (2.16) by p_i,

$$p_i\dot{q}^i = p_i(M^+)^{ij}p_j + p_i(M^+)^{ij}A_j + \lambda^a u_a^i p_i. \tag{2.18}$$

Using these equations, the Legendre transform is non-degenerate and directly yields the total Hamiltonian,

$$H_T = H_C + \lambda^a \mathcal{P}_a, \tag{2.19a}$$
$$H_C = \tfrac{1}{2}(p_i + A_i)(M^+)^{ij}(p_j + A_j) + V. \tag{2.19b}$$

where H_C is the canonical Hamiltonian and the functions,

$$\mathcal{P}_a = u_a^i(p_i + A_i), \tag{2.20}$$

are enforced to vanish by the Lagrange multipliers λ^a, so that, $\mathcal{P}_a = 0$ yields so-called *primary constraints*. These also follow directly from contracting (2.11) with u_a^i.

The existence of primary constraints, which we saw followed from the non-invertibility of M_{ij}, implies that not all variables are dynamical and that there exist non-dynamical equations of motion (2.6). We will call a coordinate q^i *dynamical* if it has an independent conjugate momentum not determined by a primary constraint. Otherwise, q^i is non-dynamical. Equations of motion, or linear combinations thereof, are called non-dynamical if they contain no time derivatives of the conjugate momenta, and such relations are by definition constraints. If one can identify non-dynamical relations that solve for q^i and its conjugate momentum p_i in terms of the remaining variables, then q^i and p_i are *non-propagating*.

The equations of motion can now conveniently be formulated by *Hamilton's equations*,

$$\dot{q}^i \approx \frac{\partial H_T}{\partial p_i}, \tag{2.21a}$$
$$\dot{p}_i \approx -\frac{\partial H_T}{\partial q^i}, \tag{2.21b}$$

where we have used the notation of *weak equalities* $\approx$ to emphasise that these equations are to be evaluated on the constraint surface.

It is convenient to introduce the *Poisson brackets*,

$$\{A(q^i, p_i), B(q^j, p_j)\} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}, \tag{2.22}$$

which can be used to write Hamilton's equations,

$$\dot{q}^i \approx \{q^i, H_T\}, \qquad \dot{p}_i \approx \{p_i, H_T\}. \tag{2.23}$$

Note that for any function $A(q^i, p_i)$, its time derivative can be expanded,

$$\dot{A} = \frac{\partial A}{\partial q^i}\dot{q}^i + \frac{\partial A}{\partial p_i}\dot{p}_i, \tag{2.24}$$

which if we substitute (2.21a) reads,

$$\dot{A} = \frac{\partial A}{\partial q^i}\frac{\partial H_T}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H_T}{\partial q^i} = \{A, H_T\}, \tag{2.25}$$

so that evaluating the time derivatives of a phase-space function via the bracket with H_T, $\dot{A} = \{A, H_T\}$ automatically substitutes Hamilton's equations and is thus an algebraic expression.

The primary constraints are obtained by differentiating the total Hamiltonian with respect to the arbitrary Lagrange multipliers λ^a,

$$\frac{\partial H_T}{\partial \lambda^a} = \mathcal{P}_a \approx 0, \tag{2.26}$$

and since the primary constraints $\mathcal{P}_a = 0$ hold at all times, their time derivatives must vanish,

$$\dot{\mathcal{P}}_a = \{\mathcal{P}_a, H_T\} \approx 0. \tag{2.27}$$

In contrast to explicitly computing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_a$, which generically is a function of $\dot{p}_i$ and $\dot{q}^i$ and vanishes when $\mathcal{P}_a = 0$, the corresponding conditions expressed in terms of Poisson brackets are *not automatic*, even on the primary constraint surface. As noted above, the bracket with H_T substitutes $\dot{p}_i$ and $\dot{q}^i$ for their equations of motion (2.21a), so these conditions instead constitute a linear combination of the algebraic (2.21a), which might or might not trivially vanish on the primary constraint surface.

If $\dot{\mathcal{P}}_a \approx 0$ trivially, it yields no new conditions, if not, it generates *secondary constraints*, which are precisely the Lagrangian conditions (2.6) expressed in terms of canonical variables,

$$\mathcal{C}_a^{(2)} = \dot{\mathcal{P}}_a \approx 0. \tag{2.28}$$

Since all time derivatives of $\mathcal{P}_a$ must vanish, $\dot{\mathcal{P}}_a = \ddot{\mathcal{P}}_a = \ldots = 0$, the secondary constraints $\mathcal{C}^{(2)} \approx 0$ must be preserved in time, so we impose,

$$\dot{\mathcal{C}}_a^{(2)} = \{\mathcal{C}_a^{(2)}, H_T\} = \{\mathcal{C}_a^{(2)}, H_C\} + \lambda^b\{\mathcal{C}_a^{(2)}, \mathcal{P}_b\} \approx 0. \tag{2.29}$$

These conditions either vanish on the constraint surface defined by $\mathcal{P}_a = 0$ and $\mathcal{C}_a^{(2)} = 0$, or they are non-trivial. If they are non-trivial and the bracket matrix $m_{ab} = \{\mathcal{C}_a^{(2)}, \mathcal{P}_b\}$ is invertible, the equations determine all Lagrange multipliers. However, if m_{ab} is not invertible, some linear combinations of (2.29) are independent of λ^a. These combinations yield *tertiary constraints*, $\mathcal{C}_\alpha^{(3)} = 0$ which constrain the canonical variables. The process of imposing the stability $\{\mathcal{C}^{(n)}, H_T\} \approx 0$ continues until, either all Lagrange multipliers are determined, or imposing $\dot{\mathcal{C}}^{(n)} \approx 0$ does not yield any new constraints. This so-called *Dirac–Bergmann algorithm* [47–49] is a procedural way of obtaining all the constraints of a theory.

It is worth noting that if a constraint determines a non-dynamical variable Q, the stability of that constraint will not yield any additional constraints but instead determine the Lagrange multiplier λ_Q multiplying the momenta conjugate to Q. E.g. if $\mathcal{C}(q, \dot{q}, Q) = 0$ can be solved for $Q = f(q, \dot{q})$, then the constraint can equivalently be written locally as $\mathcal{C}' = Q - f(q, \dot{q}) = 0$, and its stability reads,

$$\dot{\mathcal{C}}' = \{\mathcal{C}', H_C\} - \lambda^a \{f, \mathcal{P}_a\} + \lambda_Q \approx 0, \tag{2.30}$$

determining $\lambda_Q \approx -\{\mathcal{C}', H_C\} + \lambda^a \{f, \mathcal{P}_a\}$.

2.3 Classification of Constraints

Aside from the step of the algorithm at which a constraint arises, there is no intrinsic distinction between primary, secondary, tertiary, ... constraints. A meaningful classification is instead given by first- and second-class constraints. A constraint is called *first class* if its Poisson bracket with every constraint vanishes weakly. That is, if $\{\phi_A\}$ denotes the set of all constraints, then $\varphi(q^i, p_i)$ is first class provided,

$$\{\varphi, \phi_A\} \approx 0, \tag{2.31}$$

for all ϕ_A. A constraint that is not first class is called *second class*. The notion of primary, secondary, ... constraints is independent of them being first or second class, e.g. a primary constraint can be either first or second class.

It can be shown that the Poisson bracket of two first-class constraints is again first-class [49, 50]. Moreover, if $\{\varphi_\alpha\}$ denotes the set of first-class constraints, their brackets close,

$$\{\varphi_\alpha, \varphi_\beta\} = C_{\alpha\beta}{}^{\gamma}(q, p)\, \varphi_\gamma, \tag{2.32}$$

for some structure functions $C_{\alpha\beta}{}^{\gamma}(q, p)$. Such a first-class *constraint algebra* is generically associated with gauge symmetries of the system,[3] in the sense that

[3] There are, however, counterexamples: first-class constraints do not imply a gauge symmetry in full generality. For the class of regular theories considered here, one can nevertheless show that the first-class constraints generate the gauge transformations, so the corresponding first-class algebra reflects the gauge symmetry of the system [50].

one may construct a *gauge generator* $G[\xi]$ as a suitable linear combination of first-class constraints, with gauge parameters ξ as coefficients. $G[\xi]$ then generates infinitesimal gauge transformations via the Poisson bracket, and the fact that the theory is invariant under the gauge symmetry means that,

$$\{G[\xi], H_T\} \approx 0, \tag{2.33}$$

and that solutions $q^i(t)$ are mapped to solutions, so that,

$$\delta_\xi q^i = \{q^i, G[\xi]\}, \tag{2.34}$$

is a pure gauge or symmetry displacement.[4]

The symmetry generators can be constructed via a *Castellani chain* [51],[5]

$$G[\xi] = \sum_{n=0}^{k} \frac{\mathrm{d}^n \xi(t)}{\mathrm{d}t^n} G_n, \tag{2.35}$$

where each G_n is first-class and the last term G_k is primary. Given the primary first-class constraint G_k, the rest of the chain can be constructed iteratively,

$$G_{n-1} = -\{G_n, H_T\} + (\text{primary constraints}), \tag{2.36}$$

and the chain by construction terminates at G_0.

Since the primary first-class constraint G_k has vanishing Poisson bracket with all other constraints ϕ^A, the Dirac–Bergmann algorithm will never determine the Lagrange multiplier λ_G. Since every step in the Dirac–Bergmann algorithm has the form,

$$\{\phi^A, H_T\} = \{\phi^A, H_C\} + \lambda^a\{\phi^A, \mathcal{P}_a\} + \lambda_G\{\phi^A, G_k\} \approx 0, \tag{2.37}$$

and the last bracket is necessarily weakly zero, every step will be independent of λ_G.

This means that some coordinates and momenta will have equations of motion containing Lagrange multipliers that are not determined by the theory, and the solutions thus depend on arbitrary functions of time. Alternatively, one can gauge fix either by making a choice of the affected variables or the undetermined Lagrange multipliers (effectively gauge fixing the time-derivative of the affected variables). This additional redundancy means that every first-class constraint, apart from the constraint equation itself $\varphi_\alpha \approx 0$, eliminates one additional degree of freedom in phase space.

[4]Note that it is generically not possible to construct $G[\xi]$ so that it strongly generates the symmetry transformation for the momenta [51].

[5]In field theory, ξ generically depends on both space and time, and the G are smeared operators.

When all constraints are obtained and classified one can compute the dimensionality of the physical phase space using the formula,

$$2\times\begin{pmatrix}\text{Number of}\\ \text{physical degrees}\\ \text{of freedom}\end{pmatrix} = \begin{pmatrix}\text{Number of}\\ \text{phase-space}\\ \text{dimensions}\end{pmatrix} - 2\times\begin{pmatrix}\text{Number of}\\ \text{first-class}\\ \text{constraints}\end{pmatrix} - \begin{pmatrix}\text{Number of}\\ \text{second-class}\\ \text{constraints}\end{pmatrix}. \tag{2.38}$$

For phase space to be even dimensional, the second-class constraints must always come in pairs.

Chapter 3

General Relativity

This chapter first reviews the covariant formulation of GR, the metric tensor, covariant derivative, curvature, the Einstein–Hilbert action, and Einstein's equations, highlighting the presence of constraints and diffeomorphism gauge symmetry. We then review the canonical Hamiltonian formulation via a 3+1 decomposition of the metric and identify the constraints and their algebra using the Dirac–Bergmann algorithm. Finally, we introduce the canonical vielbein formulation using a 3+1 decomposition of the vielbein, and review the emergence of the rotational primary constraints from the degenerate Legendre transform, the resulting constraint algebra, and the extension of phase space that restores full $\mathrm{SO}(1,3)$ invariance by explicitly constructing the rotation and boost generators.

3.1 Covariant Formulation

Einstein's theory of General Relativity is formulated in terms of a dynamical spacetime, i.e. a four-dimensional Lorentzian manifold whose geometry is encoded in a *metric tensor*,

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{3.1}$$

often denoted by its components $g_{\mu\nu}$.

The curvature of the manifold is described by the *Riemann curvature* tensor $R^\mu{}_{\nu\alpha\beta}$, defined through its action on an arbitrary spacetime vector field $X^\nu(x)$ by,

$$R^\mu{}_{\nu\alpha\beta}X^\nu = \Big[\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\Big]X^\mu, \tag{3.2}$$

where ∇_ν is the torsion-free, metric-compatible *covariant derivative*, uniquely determined by the condition $\nabla_\alpha g_{\mu\nu} = 0$. The covariant derivative can be defined by its action on a spacetime vector X^ν,

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\alpha}X^\alpha, \tag{3.3}$$

where ${\Gamma^\nu}_{\mu\alpha}$ are functions of $g_{\mu\nu}$ called *Christoffel symbols*, and are determined entirely in terms of the metric components,

$$ {\Gamma^\mu}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}\Big[\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\Big], \tag{3.4} $$

where $g^{\mu\nu}$ denotes the inverse metric $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$.

The action of ∇_μ on scalars is defined by $\nabla_\mu \phi = \partial_\mu \phi$, which can be used to generalise its action on covectors and tensor densities. For example, for a tensor density $\mathfrak{X}^\nu$ of weight one,[1]

$$ \nabla_\mu \mathfrak{X}^\nu = \partial_\mu \mathfrak{X}^\nu + {\Gamma^\nu}_{\mu\sigma}\mathfrak{X}^\sigma - {\Gamma^\sigma}_{\mu\sigma}\mathfrak{X}^\nu. \tag{3.5} $$

Combining (3.2) and (3.4), the Riemann tensor can be determined in terms of the metric field,

$$ {R^\mu}_{\nu\alpha\beta} = \partial_\alpha {\Gamma^\mu}_{\beta\nu} - \partial_\beta {\Gamma^\mu}_{\alpha\nu} + {\Gamma^\mu}_{\alpha\sigma}{\Gamma^\sigma}_{\beta\nu} - {\Gamma^\mu}_{\beta\sigma}{\Gamma^\sigma}_{\alpha\nu}. \tag{3.6} $$

From the Riemann tensor one can construct the *Ricci tensor* $R_{\mu\nu}$ and *Ricci scalar* R,

$$ R_{\mu\nu} = {R^\alpha}_{\mu\alpha\nu}, \tag{3.7a} $$
$$ R = g^{\mu\nu} R_{\mu\nu}. \tag{3.7b} $$

The dynamics of the metric can be derived using the principle of stationary action (2.2) with the gravitational action,

$$ \mathcal{S}[g_{\mu\nu}, \psi_a] = \mathcal{S}_{\text{EH}}[g_{\mu\nu}] + \mathcal{S}_{\text{m}}[g_{\mu\nu}, \psi_a], \tag{3.8} $$

where $\mathcal{S}_{\text{EH}}[g_{\mu\nu}]$ is the *Einstein–Hilbert action*,

$$ \mathcal{S}_{\text{EH}}[g_{\mu\nu}] = m_{\text{pl}}^2 \int \mathrm{d}^4x \sqrt{-g}\Big[R(g) - 2\Lambda\Big], \tag{3.9} $$

built from $\sqrt{-g} = (-\det g_{\mu\nu})^{1/2}$, the Ricci scalar (3.7b), and a *cosmological constant* Λ. The coupling to matter fields ψ_a is given in terms of $\mathcal{S}_{\text{m}}$, and the coupling strength set by the Planck mass m_{pl}^2.

Varying the gravitational action with respect to $g^{\mu\nu}$ yields,[2]

$$ \frac{\delta \mathcal{S}_{\text{EH}}}{\delta g^{\mu\nu}} = m_{\text{pl}}^2 \sqrt{-g}\,\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right), \tag{3.10a} $$
$$ \frac{\delta \mathcal{S}_{\text{m}}}{\delta g^{\mu\nu}} = -\tfrac{1}{2}\sqrt{-g}\, T_{\mu\nu}, \tag{3.10b} $$

[1] i.e. $\mathfrak{X}^\nu/\sqrt{-g}$ is a tensor.

[2] The variation of (3.9) produces boundary terms involving normal derivatives of the metric variations $\delta g_{\mu\nu}$, which do not vanish by imposing only the initial data on the boundary $\delta g_{\mu\nu}\big|_{\partial\mathcal{M}} = 0$. A well-posed variational principle can be restored by adding the Gibbons–Hawking–York boundary term [52, 53]. We assume throughout that such boundary terms are included.

where,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R, \tag{3.11}$$

is the *Einstein tensor* and $T_{\mu\nu}$ is the *stress-energy tensor*. Einstein's field equations then take the form,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2m_{\rm pl}^2} T_{\mu\nu}. \tag{3.12}$$

A useful observation, already at the level of the field equations (3.12), is that General Relativity is a constrained system. For example, from the contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ it follows that,

$$\partial_0 G^{0\nu} = -\partial_i G^{i\nu} - \Gamma^{\alpha}{}_{\alpha\sigma} G^{\sigma\nu} - \Gamma^{\nu}{}_{\alpha\sigma} G^{\alpha\sigma}, \tag{3.13}$$

and since the right-hand side contains at most second time derivatives, $G^{0\nu}$ can involve at most first time derivatives. This implies that the (0ν) components of Einstein's equations (3.12),

$$G_{0\nu} + \Lambda g_{0\nu} = \frac{1}{2m_{\rm pl}^2} T_{0\nu}, \tag{3.14}$$

are first order in time, and therefore impose constraints on the initial data rather than providing dynamical equations.

We will see that these constraints are a result of a gauge symmetry associated with diffeomorphism invariance. The gravitational action is by construction diffeomorphism invariant. That is, given an infinitesimal diffeomorphism parametrised by a vector field $\xi^\mu(x)$, the Lagrangian, which is a scalar density, transforms as,

$$\delta_\xi \mathcal{L} = \partial_\mu(\xi^\mu \mathcal{L}), \tag{3.15}$$

so that the action transforms only up to a boundary term,

$$\delta \mathcal{S}_\xi = \int \mathrm{d}^4x \, \delta_\xi \mathcal{L} = \int \mathrm{d}^4x \, \partial_\mu(\xi^\mu \mathcal{L}) = 0, \tag{3.16}$$

which vanishes since ξ^μ is zero on the boundary.

Under the same transformation, the metric $g_{\mu\nu}$ and matter fields ψ_a transform as,

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}, \tag{3.17a}$$

$$\delta_\xi \psi_a = \mathcal{L}_\xi \psi_a = \xi^\mu \nabla_\mu \psi_a + \Delta_a{}^{\mu\nu} \nabla_\mu \xi_\nu, \tag{3.17b}$$

where the details of $\Delta_a{}^{\mu\nu}$ depend on the tensorial/spinor structure of the matter field. These transformations induce the variation of the gravitational action,

$$\delta_\xi \mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\Big[-2m_{\rm pl}^2\Big(G^{\mu\nu} + \Lambda g^{\mu\nu} - \tfrac{1}{2m_{\rm pl}^2}T^{\mu\nu}\Big)\nabla_\mu\xi_\nu + E^a\Big(\xi^\mu\nabla_\mu\psi_a + \Delta_a{}^{\mu\nu}\nabla_\mu\xi_\nu\Big)\Big] = 0, \tag{3.18}$$

which by integration by parts of the first and last terms, yields a non-trivial relation between the field equations and their derivatives which vanishes identically,[3]

$$2m_{\rm pl}^2\,\nabla_\mu\Big(G^{\mu\nu} + \Lambda g^{\mu\nu} - \tfrac{1}{2m_{\rm pl}^2}T^{\mu\nu}\Big) - \nabla_\mu(E^a\Delta_a{}^{\mu\nu}) + E^a\nabla^\nu\psi_a = 0. \tag{3.19}$$

In Section 2.3, we argued that this is the hallmark of a *gauge symmetry*, and we thus identify diffeomorphism invariance as the gauge symmetry of General Relativity, and thus expect the existence of first-class constraints.

Note that the first two terms in (3.19) vanish identically by the Bianchi identity and metric compatibility. Imposing the matter field equations $E^a = 0$ then yields,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.20}$$

i.e. on-shell local conservation of the energy-momentum tensor.

3.2 Canonical Metric Formulation

Einstein's field equations can be formulated in Hamiltonian form. This can be useful for numerical evolution schemes, a systematic analysis of the constraint structure or even canonical quantisation. In contrast to the Lagrangian formulation, the Hamiltonian formulation singles out time and therefore treats space and time differently, whereas the standard covariant formulation of General Relativity is manifestly spacetime symmetric.

One way to obtain a canonical formulation is to assume the spacetime admits a *foliation* by spacelike hypersurfaces Σ_t and perform a so-called *3+1 decomposition* [54, 55],[4] so that the spacetime manifold is a product $\mathcal{M} = \mathbb{R}\times\Sigma_t$. Each Σ_t is a spatial hypersurface and parametrised by a time parameter t.

This is usually done by the choice of a *timelike unit vector* n^μ ($g_{\mu\nu}n^\mu n^\nu = -1$), which is everywhere normal to the hypersurface Σ_t, and can be used to construct the spatial metric,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{3.21}$$

[3] The $\nabla_\mu T^{\mu\nu}$ and $\nabla_\mu(E^a\Delta_a{}^{\mu\nu})$ terms can conveniently be written in terms of the Belinfante–Rosenfeld stress-energy $T^{\mu\nu}_{\rm BR} = T^{\mu\nu} + E^a\Delta_a^{\mu\nu}$.

[4] While ADM (Arnowitt, Deser and Misner) is a standard reference for Hamiltonian 3+1 treatment of General Relativity, the formalism was developed prior to their papers. The work by Rosenfeld [56], Bergmann [47], Anderson [57], and Dirac [58, 59], which laid the groundwork for ADM is often overlooked.

If the choice of coordinates is adjusted to the foliation, the spacetime metric $g_{\mu\nu}$ can be parametrised in terms of a *lapse function* $N(x)$, a *shift vector* $N^i(x)$, and an induced 3-dimensional *Riemannian metric* $\gamma_{ij}(x)$ on Σ_t,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\big(\mathrm{d}x^i + N^i\mathrm{d}t\big)\big(\mathrm{d}x^j + N^j\mathrm{d}t\big), \tag{3.22}$$

or in component block matrix notation,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & \gamma_{ij} \end{pmatrix}, \tag{3.23}$$

where $N_i = \gamma_{ij}N^j$. The inverse metric can also be expressed in the *3+1 variables*, N, N^i, γ_{ij},

$$g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & N^j \\ N^i & N^2\gamma^{ij} - N^i N^j \end{pmatrix}, \tag{3.24}$$

where γ^{ij} is the inverse of γ_{ij}, so that $\gamma^{ik}\gamma_{kj} = \delta^i_j$. The indices of tensors on Σ_t are raised and lowered with γ^{ij} and γ_{ij} respectively. The normal vector n^μ conveniently takes the form,

$$n^\mu = (1/N, -N^i/N), \tag{3.25}$$

in terms of the 3+1 variables.

The Ricci curvature of the spatial hypersurfaces can be related to the spacetime curvature, via the *Gauss-Codazzi relation*,

$$R = {}^{(3)}R + K^{ij}K_{ij} - K^2 - 2\nabla_\mu v^\mu, \tag{3.26}$$

where,

$$v^\mu = n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu, \tag{3.27}$$

and the *extrinsic curvature* K_{ij} is defined by the Lie derivative of the spatial metric along the unit normal vector to the hypersurface n^μ,[5]

$$K_{ij} = \tfrac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ij} = \frac{1}{2N}\Big[\dot{\gamma}_{ij} - 2\,{}^{(3)}\nabla_{(i}N_{j)}\Big], \qquad K = \gamma^{ij}K_{ij}. \tag{3.28}$$

${}^{(3)}\nabla_i$ is the torsion-free covariant derivative compatible with the spatial metric γ_{ij} with corresponding Christoffel symbols ${}^{(3)}\Gamma^i{}_{jk}$ and spatial curvature tensors

[5] The Lie derivative $\mathcal{L}_{\vec{\xi}}$ along a vector field ξ^i measures the change of a tensor field along the flow generated by ξ^i. For a spatial rank-$(0,2)$ tensor h_{ij} it is given by,

$$\mathcal{L}_{\vec{\xi}}h_{ij} = \xi^k\partial_k h_{ij} + h_{kj}\partial_i\xi^k + h_{ik}\partial_j\xi^k.$$

${}^{(3)}R^i{}_{jkl}$, ${}^{(3)}R_{ij}$, and ${}^{(3)}R$, constructed by equations analogous to (3.4–3.7). We emphasise that the spatial curvatures are constructed from γ_{ij} and its spatial derivatives only.

Using (3.26), the Einstein–Hilbert action (3.9) takes the form,

$$\mathcal{S}_{\text{EH}} = m_{\text{pl}}^2 \int \mathrm{d}^4x\, N\sqrt{\gamma}\Big[\Big({}^{(3)}R - 2\Lambda\Big) + K^{ij}K_{ij} - K^2 - 2\nabla_\mu v^\mu\Big]. \tag{3.29}$$

The last term is a boundary term $\int \mathrm{d}^4x\, \partial_\mu\big(\sqrt{-g}\,v^\mu\big)$ which we will disregard.[6] This makes the action independent of time derivatives of the lapse N and shift N^i, so that when we introduce canonical momenta conjugate to the fields N, N^i, and γ_{ij},

$$P = \frac{\partial \mathcal{L}_{\text{EH}}}{\partial \dot{N}}, \qquad P_i = \frac{\partial \mathcal{L}_{\text{EH}}}{\partial \dot{N}^i}, \qquad \pi^{ij} = \frac{\partial \mathcal{L}_{\text{EH}}}{\partial \dot{\gamma}_{ij}}. \tag{3.30}$$

P and P_i vanish trivially, generating primary constraints,

$$P = 0, \qquad P_i = 0. \tag{3.31}$$

The metric momenta π^{ij} are related to the velocities $\dot{\gamma}_{ij}$ by,

$$\pi^{ij} = M^{ijkl}\Big[\dot{\gamma}_{kl} - 2\,{}^{(3)}\nabla_k N_l\Big], \tag{3.32a}$$

$$M^{ijkl} = \tfrac{m_{\text{pl}}^2\sqrt{\gamma}}{2N}\Big[\gamma^{i(k}\gamma^{l)j} - \gamma^{ij}\gamma^{kl}\Big]. \tag{3.32b}$$

M^{ijkl} can be explicitly inverted on the space of symmetric rank-2 tensors,

$$\big(M^{-1}\big)_{ijkl} = \tfrac{2N}{m_{\text{pl}}^2\sqrt{\gamma}}\Big[\gamma_{i(k}\gamma_{l)j} - \tfrac{1}{2}\gamma_{ij}\gamma_{kl}\Big], \tag{3.33a}$$

$$\big(M^{-1}M\big)^{ij}{}_{kl} = \delta^i_{(k}\delta^j_{l)}. \tag{3.33b}$$

So that the inversion for $\dot{\gamma}_{ij}$ in terms of π^{ij} takes the form,

$$\begin{aligned}\dot{\gamma}_{ij} &= (M^{-1})_{ijkl}\pi^{kl} + 2^{(3)}\nabla_{(i}N_{j)} \\ &= \tfrac{2N}{m_{\text{pl}}^2\sqrt{\gamma}}\Big[\pi_{ij} - \tfrac{1}{2}\gamma_{ij}\pi^k{}_k\Big] + 2\,{}^{(3)}\nabla_{(i}N_{j)}.\end{aligned} \tag{3.34}$$

Adding the primary constraints $P = 0$ and $P_i = 0$ with associated Lagrange multipliers λ and λ^i, the Legendre transform yields a Hamiltonian linear in lapses N and shifts N^i,

$$H_T = -\int \mathrm{d}^3x\Big[N\mathcal{H} + N^i\mathcal{H}_i + \lambda P + \lambda^i P_i\Big], \tag{3.35}$$

[6] Discarding this "temporal York–Gibbons–Hawking" boundary term removes the second time derivatives of γ_{ij} together with all time derivatives of N and N^i from the action.

so that the lapse and shift effectively become Lagrange multipliers for the conditions $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ defined by,

$$\mathcal{H} = m_{\text{pl}}^2\sqrt{\gamma}\left[{}^{(3)}R - 2\Lambda\right] + \tfrac{1}{m_{\text{pl}}^2\sqrt{\gamma}}\left[\tfrac{1}{2}(\pi^i{}_i)^2 - \pi^{ij}\pi_{ij}\right], \tag{3.36a}$$

$$\mathcal{H}_i = 2\gamma_{ij}{}^{(3)}\nabla_k\pi^{jk}. \tag{3.36b}$$

The presence of $\sqrt{\gamma}$ in (3.32) makes π^{ij} into a tensor density of weight 1, so that (3.36b) must be computed using (3.5),

$${}^{(3)}\nabla_k\pi^{ij} = \partial_k\pi^{ij} + {}^{(3)}\Gamma^i{}_{kl}\pi^{lj} + {}^{(3)}\Gamma^j{}_{kl}\pi^{il} - {}^{(3)}\Gamma^l{}_{kl}\pi^{ij}. \tag{3.37}$$

Similarly, $\mathcal{H}$ and $\mathcal{H}_i$ are densities of weight 1.

We now introduce the non-vanishing *canonical Poisson brackets*,

$$\{N(x), P(y)\} = \delta(x-y), \tag{3.38a}$$

$$\{N^i(x), P_j(y)\} = \delta^i_j\delta(x-y), \tag{3.38b}$$

$$\{\gamma_{ij}(x), \pi^{kl}(y)\} = \delta_i^{(k}\delta_j^{l)}\delta(x-y), \tag{3.38c}$$

where $\delta(x-y)$ is the 3-dimensional delta function. With the canonical structure provided by (3.38), we can compute Poisson brackets of arbitrary phase-space functions $A(x)$ and $B(x)$ by,

$$\{A(x), B(y)\} = \sum_I \int \mathrm{d}^3z\left[\frac{\delta A(x)}{\delta\mathcal{Q}^I(z)}\frac{\delta B(y)}{\delta\Pi_I(z)} - \frac{\delta B(y)}{\delta\mathcal{Q}^I(z)}\frac{\delta A(x)}{\delta\Pi_I(z)}\right], \tag{3.39}$$

with the generalised coordinates $\mathcal{Q}^I = (N, N^i, \gamma_{ij})$ and conjugate momenta $\Pi_I = (P, P_i, \pi^{ij})$.

The Dirac–Bergmann algorithm requires us to impose the preservation in time of the primary constraints (3.31). This yields the *Hamiltonian* and *momentum constraints*,

$$\begin{aligned}\dot{P}(x) &= \{P(x), H_T\} \approx \mathcal{H}(x) \approx 0,\\ \dot{P}_i(x) &= \{P_i(x), H_T\} \approx \mathcal{H}_i(x) \approx 0.\end{aligned} \tag{3.40}$$

Since these secondary constraints are non-trivial and independent of Lagrange multipliers, one needs to impose their stability,

$$\dot{\mathcal{H}}(x) \approx -\int \mathrm{d}^3y\left[N(y)\{\mathcal{H}(x)\,,\mathcal{H}(y)\} + N^j(y)\{\mathcal{H}(x)\,,\mathcal{H}_j(y)\}\right] \approx 0, \tag{3.41}$$

$$\dot{\mathcal{H}}_i(x) \approx -\int \mathrm{d}^3y\left[N(y)\{\mathcal{H}_i(x), \mathcal{H}(y)\} + N^j(y)\{\mathcal{H}_i(x), \mathcal{H}_j(y)\}\right] \approx 0. \tag{3.42}$$

These brackets can be explicitly computed [60],[7]

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = \Big[\mathcal{H}^i(y)\frac{\partial}{\partial y^i} - \mathcal{H}^i(x)\frac{\partial}{\partial x^i}\Big]\delta(x-y), \tag{3.43a}$$

$$\{\mathcal{H}(x), \mathcal{H}_i(y)\} = \mathcal{H}(y)\frac{\partial}{\partial y^i}\delta(x-y), \tag{3.43b}$$

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \Big[\mathcal{H}_i(x)\frac{\partial}{\partial y^j} - \mathcal{H}_j(y)\frac{\partial}{\partial x^i}\Big]\delta(x-y), \tag{3.43c}$$

where $\mathcal{H}^i = \gamma^{ij}\mathcal{H}_j$. These are all proportional to the secondary constraints themselves and thus vanish weakly, not generating any further constraints. Since $\mathcal{H}$ and $\mathcal{H}_i$ are independent of lapse and shift, we also trivially have,

$$\{P(x), \mathcal{H}(y)\} = 0, \qquad \{P(x), \mathcal{H}_i(y)\} = 0, \tag{3.44a}$$

$$\{P_i(x), \mathcal{H}(y)\} = 0, \qquad \{P_i(x), \mathcal{H}_j(y)\} = 0, \tag{3.44b}$$

making all brackets between the functions $P, P_i, \mathcal{H}, \mathcal{H}_i$ weakly zero, implying that they are all first class. This makes (3.43–3.44) the expected first-class algebra corresponding to the diffeomorphism invariance of the Einstein–Hilbert action.

The smeared functions,

$$\mathcal{H}[\chi] = \int \mathrm{d}^3x\, \chi(x)\mathcal{H}(x), \qquad \vec{\mathcal{H}}[\xi^i] = \int \mathrm{d}^3x\, \xi^i(x)\mathcal{H}_i(x), \tag{3.45}$$

correspond to the generators of infinitesimal temporal and spatial diffeomorphisms acting on γ_{ij} and π^{ij} and are parametrised by χ and ξ^i. For example, a spatial diffeomorphism parametrised by the flow of a vector $\xi^i(x)$ is generated by $\vec{\mathcal{H}}[\xi^i]$ via the Poisson brackets,[8]

$$\{\gamma_{ij}, \vec{\mathcal{H}}[\xi^i]\} = -\mathcal{L}_{\vec{\xi}}\gamma_{ij}, \tag{3.46a}$$

$$\{\pi^{ij}, \vec{\mathcal{H}}[\xi^i]\} = -\mathcal{L}_{\vec{\xi}}\pi^{ij}, \tag{3.46b}$$

corresponding precisely to the Lie flow generated by ξ^i. The Hamiltonian constraint $\mathcal{H}[\chi]$ generates normal deformations of the spatial hypersurface corresponding to temporal diffeomorphisms.

These generators do not generate diffeomorphisms on the full phase space since both $\mathcal{H}$ and $\mathcal{H}_i$ Poisson commute with N and N^i and their momenta. So

[7] See appendix A of [61] for a pedagogic derivation of this particular form.

[8] Since π^{ij} is a tensor density,

$$\mathcal{L}_{\vec{\xi}}\pi^{ij} = \partial_k(\xi^k\pi^{ij}) - \pi^{ik}\partial_k\xi^j - \pi^{kj}\partial_k\xi^i,$$

while

$$\mathcal{L}_{\vec{\xi}}\gamma_{ij} = \xi^k\partial_k\gamma_{ij} + \gamma_{ik}\partial_j\xi^k + \gamma_{kj}\partial_i\xi^k = 2\,{}^{(3)}\nabla_{(i}\xi_{j)}.$$

to construct the full generator we can use the Castellani construction (2.35–2.36), which yields [51],

$$\begin{aligned} G[\chi] &= -\int \mathrm{d}^3x \Big[\chi\Big(\mathcal{H} + N^j\partial_j P + \partial_j(PN^j) + \partial_j(NP^j)\Big) + \dot{\chi}P\Big], \\ \vec{G}[\xi^i] &= -\int \mathrm{d}^3x \Big[\xi^i\Big(\mathcal{H}_i + \partial_i N^j P_j + \partial_j(N^j P_i) + \partial_i NP\Big) + \dot{\xi}^i P_i\Big]. \end{aligned} \tag{3.47}$$

corresponding to the full generators of the set of diffeomorphisms on all the phase-space variables.

Einstein's field equations can be formulated in terms of Hamilton's equations,

$$\dot{\gamma}_{ij} \approx \{\gamma_{ij}, H_T\} \approx 2NK_{ij} + \mathcal{L}_{\vec{N}}\gamma_{ij}, \tag{3.48a}$$

$$\dot{\pi}^{ij} \approx \{\pi^{ij}, H_T\} \approx -\{\pi^{ij}, \mathcal{H}[N]\} + \mathcal{L}_{\vec{N}}\pi^{ij}, \tag{3.48b}$$

$$\dot{N} \approx \{N\,,\, H_T\} \approx -\lambda, \tag{3.48c}$$

$$\dot{N}^i \approx \{N^i, H_T\} \approx -\lambda^i, \tag{3.48d}$$

where (3.48a) reproduces the definition of the momenta (3.34), and the equations for $\dot{N}$ and $\dot{N}^i$ determine the Lagrange multipliers, while the dynamics is contained in (3.48b), where,

$$\begin{aligned} \{\pi^{ij}, \mathcal{H}[N]\} \approx N&\Big[2K^i{}_k\delta^j_l - \tfrac{1}{2}\gamma^{ij}K_{kl}\Big]\pi^{kl} \\ &+ Nm_{\mathrm{pl}}^2\sqrt{\gamma}\Big[{}^{(3)}G^{ij} + \Lambda\gamma^{ij}\Big] \\ &+ m_{\mathrm{pl}}^2\sqrt{\gamma}\Big[\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}\Big]{}^{(3)}\nabla_k{}^{(3)}\nabla_l N, \end{aligned} \tag{3.49}$$

and ${}^{(3)}G_{ij}$ is the spatial Einstein tensor,

$${}^{(3)}G_{ij} = {}^{(3)}R_{ij} - \tfrac{1}{2}\gamma_{ij}{}^{(3)}R. \tag{3.50}$$

Hamilton's equations for P and P_i have already been computed and yielded the Hamiltonian and momentum constraints (3.40).

Computing the dimensionality of the physical phase space, and thus the number of propagating modes can now be done using (2.38). The canonical variables $(N,P),(N^i,P_i),(\gamma_{ij},\pi^{ij})$, form a 2×10 dimensional phase space, while the eight first-class constraints $P\approx 0$, $P_i\approx 0, \mathcal{H}\approx 0, \mathcal{H}_i \approx 0$ reduce this by 2×8 dimensions, yielding a 2×2 dimensional physical phase space, corresponding to the 2 helicities of a massless spin-2 field and their momenta.

3.3 Canonical Vielbein Formulation

While General Relativity is most often formulated in terms of a metric field $g_{\mu\nu}(x)$, it is sometimes convenient or even necessary, e.g. when considering spinors in curved space, to formulate the theory in terms of a vielbein $e^A{}_\mu(x)$. The *vielbein* can be implicitly defined in terms of the metric,

$$g_{\mu\nu}(x) = e^A{}_\mu(x)\eta_{AB}e^B{}_\nu(x). \tag{3.51}$$

The vielbein components $e^A{}_\mu$ form a generic invertible 4×4 matrix with 16 independent components, in contrast to the metric's 10 components. This redundancy arises from the *local Lorentz symmetry* induced by the decomposition (3.51), under which $g_{\mu\nu}$ is invariant. Explicitly, the vielbein transforms as a vector under local Lorentz transformations $L^A{}_B(x)$,

$$e^A{}_\mu(x) \mapsto L^A{}_B(x)e^B{}_\mu(x), \qquad\qquad (L^A{}_C\eta_{AB}L^B{}_D = \eta_{CD}), \tag{3.52}$$

leaving the metric (3.51) unchanged,

$$g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x), \tag{3.53}$$

implying that $e^A{}_\mu$ and $L^A{}_Be^B{}_\mu$ yield the same metric. The vielbein associated with a metric is therefore not unique. The additional six components in the vielbein correspond to *Lorentz fields*, which can be decomposed into three *boosts* and three *rotations* parametrising a local Lorentz frame.

The inverse metric $g^{\mu\nu}$ can be used to define the *inverse vielbein* $e^\mu{}_A(x)$,

$$g^{\mu\nu}(x) = e^\mu{}_A(x)\eta^{AB}e^\nu{}_B(x), \tag{3.54}$$

which can be explicitly constructed from the components $e^A{}_\mu$ and $g^{\mu\nu}$,

$$e^\mu{}_A = g^{\mu\nu}\eta_{AB}e^B{}_\nu, \tag{3.55}$$

which also corresponds to the inverse of the matrix $e^A{}_\mu$, so that, $(e^{-1})^\mu{}_A = e^\mu{}_A$. This leads to the identities,

$$e^\mu{}_Ae^A{}_\nu = \delta^\mu_\nu, \qquad\qquad e^A{}_\mu e^\mu{}_B = \delta^A_B. \tag{3.56}$$

3.3.1 Vielbein Decomposition

The vielbein $e^A{}_\mu$ can, in analogy to the metric, be decomposed in terms of the 3+1 variables. Combining the decomposed metric (3.23) and its relation to the vielbein (3.51), one can express the vielbein components $e^A{}_0$ in terms of lapses and shifts,

$$e^A{}_0 = NX^A + N^ie^A{}_i, \tag{3.57}$$

where X^A is a normalised timelike Lorentz vector orthogonal to $e^A_{\ i}$,

$$X_A X^A = -1, \qquad\qquad X_A e^A_i = 0, \tag{3.58}$$

and e^A_i is a vielbein of the spatial metric,

$$\gamma_{ij} = e^A_{\ i} \eta_{AB} e^B_{\ j}\,. \tag{3.59}$$

X^A corresponds to the unit vector n^μ (3.25) normal to the spatial hypersurface, expressed in the Lorentz frame,

$$X^A = e^A_{\ \mu} n^\mu. \tag{3.60}$$

Similar to how n^μ can be expressed in terms of lapses and shifts (3.25), X^A and thus the internal Lorentz space can be decomposed into time and space. A convenient 3+1 parametrisation of X^A is,

$$X^A = \begin{pmatrix} \alpha \\ p^a \end{pmatrix}, \qquad\qquad \alpha = \sqrt{1 + p_a p^a}, \tag{3.61}$$

which corresponds to the first column of a Lorentz boost parametrised by p^a,

$$L^A_{\ B}(p) = \begin{pmatrix} \alpha & p_b \\ p^a & A^a_{\ b} \end{pmatrix}, \qquad\qquad A^a_{\ b} = \delta^a_b + \frac{1}{1+\alpha} p^a p_b. \tag{3.62}$$

Introducing the trivial timelike unit vector $\widehat{X}^A = (1, 0)$, the boost $L^A_{\ B}(p)$ transforms $\widehat{X}^A$ to X^A,

$$L^A_{\ B} \widehat{X}^B = \begin{pmatrix} \alpha & p_b \\ p^a & A^a_{\ b} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ p^a \end{pmatrix} = X^A. \tag{3.63}$$

Factoring out the transformation $L^A_{\ B}(p)$ from e^A_i and defining $e^A_i = L^A_{\ B}(p) E^B_i$, the orthogonality condition (3.58) implies,

$$X_A e^A_i = \widehat{X}_A E^A_i = -E^0_{\ i} = 0, \tag{3.64}$$

making E^A_i block diagonal with the only non-zero components $E^a_{\ i}$.

Expressing e^A_0 (3.57) in terms of $\widehat{X}^A$, E^a_i and $L^A_{\ B}(p)$, the vielbein $e^A_{\ \mu}$ can then be written as,

$$e^A_{\ \mu} = \begin{pmatrix} \alpha & p_b \\ p^a & A^a_{\ b} \end{pmatrix} \begin{pmatrix} N & 0 \\ E^b_{\ j} N^j & E^b_{\ i} \end{pmatrix} = \begin{pmatrix} \alpha N + p_b E^b_{\ j} N^j & p_b E^b_{\ i} \\ N p^a + A^a_{\ b} E^b_{\ j} N^j & A^a_{\ b} E^b_{\ i} \end{pmatrix}. \tag{3.65}$$

$E^a_{\ i}$ is a vielbein of the spatial metric,

$$\gamma_{ij}(x) = E^a_{\ i}(x) \delta_{ab} E^b_{\ j}(x), \tag{3.66}$$

which, in contrast to (3.59), is only invariant under local spatial rotations,

$$E^a_{\ i}(x) \mapsto \Omega^a_{\ b}(x) E^b_{\ i}(x), \qquad\qquad (\Omega^a_{\ c} \delta_{ab} \Omega^b_{\ d} = \delta_{cd}). \tag{3.67}$$

The inverse vielbein can similarly be decomposed into the 3+1 variables,

$$e^\mu_{\ A} = \frac{1}{N} \begin{pmatrix} 1 & 0 \\ -N^i & N E^i_{\ b} \end{pmatrix} \begin{pmatrix} \alpha & -p_a \\ -p^b & A^b_{\ a} \end{pmatrix}. \tag{3.68}$$

3.3.2 Vielbein Hamiltonian

Substituting (3.66) into the extrinsic curvature (3.28),

$$K_{ij} = \frac{1}{N}\Big[E^a{}_{(i}\delta_{ab}\dot{E}^b{}_{j)} - {}^{(3)}\nabla_{(i}N_{j)}\Big], \tag{3.69}$$

we can express the 3+1 decomposed Einstein–Hilbert action (3.29) in terms of the variables $E^a{}_i$, p^a, N and N^i, which acquires an induced SO(3) symmetry.

We now define the momenta conjugate to $E^a{}_i$,

$$\pi^i{}_a = \frac{\partial \mathcal{L}_{\text{EH}}}{\partial \dot{E}^a{}_i} = M^{ij}{}_{ab}\Big[\dot{E}^b{}_j - E^b{}_k{}^{(3)}\nabla_j N^k\Big], \tag{3.70a}$$

$$M^{ij}{}_{ab} = \tfrac{2m_{\text{pl}}^2\sqrt{\gamma}}{N}E_{ak}\,E_{bl}\Big[\gamma^{i(l}\gamma^{j)k} - \gamma^{il}\gamma^{kj}\Big]. \tag{3.70b}$$

In contrast to the metric formulation (3.32), this is no longer uniquely invertible for the velocities $\dot{E}^a_i$ due to the non-trivial kernel of $M^{ij}{}_{ab}$. The kernel of M is spanned by $\lambda^b{}_c E^c{}_j$,

$$M^{ij}{}_{ab}\lambda^b{}_c E^c{}_j = 0, \tag{3.71}$$

for an arbitrary $\lambda_{bc} = -\lambda_{cb}$ corresponding to an infinitesimal rotation. Constructing the pseudoinverse (2.12),

$$(M^+)^{ab}{}_{ij} = \tfrac{N}{4m_{\text{pl}}^2\sqrt{\gamma}}\Big[\gamma_{ij}\delta^{ab} + E^a{}_j E^b{}_i - E^a{}_i E^b{}_j\Big], \tag{3.72}$$

to solve for $\dot{E}^a_i$ we get the generic solution by multiplying (3.70a) by M^+ and adding an arbitrary combination of the kernel,

$$\dot{E}^a_i = (M^+)^{ab}{}_{ij}\pi^j{}_b + E^a_j{}^{(3)}\nabla_i N^j + \lambda'^a_b E^b{}_i. \tag{3.73}$$

Substituting this into the Einstein–Hilbert Lagrangian and constructing the canonical one-form $\pi^i{}_a\dot{E}^a{}_i$, the Legendre transform yields the total Hamiltonian,[9]

$$H_T = -\int \mathrm{d}^3x\Big[N\mathcal{R} + N^i\mathcal{R}_i + \lambda P + \lambda^i P_i + \lambda_{ab}\mathcal{J}^{ab}\Big], \tag{3.74}$$

where we have added the primary constraints $P = 0$ and $P_i = 0$, corresponding to the vanishing of the momenta of N and N^i, (3.30) and defined the functions,

$$\mathcal{R} = m_{\text{pl}}^2\sqrt{\gamma}\Big[{}^{(3)}R - 2\Lambda\Big] + \tfrac{1}{4m_{\text{pl}}^2\sqrt{\gamma}}\Big[\tfrac{1}{2}\big(\pi^i{}_a E^a{}_i\big)^2 - \pi^i{}_a E^a{}_j \pi^j{}_b E^b{}_i\Big], \tag{3.75a}$$

$$\mathcal{R}_i = E^a{}_i{}^{(3)}\mathcal{D}_j\pi^j{}_a + \omega_{iab}\,\mathcal{J}^{ab}, \tag{3.75b}$$

$$\mathcal{J}^{ab} = \pi^{i[a}E^{b]}{}_i. \tag{3.75c}$$

[9] We have here absorbed terms proportional to $\mathcal{J}^{ab}$ into the Lagrange multiplier λ_{ab}.

Here ${}^{(3)}\mathcal{D}_i$ denotes the SO(3) covariant derivative compatible with $E^a{}_i$ (${}^{(3)}\mathcal{D}_i E^a{}_j = 0$) with associated spin-connection $\omega_{iab} = E^j{}_{[a}{}^{(3)}\nabla_i E_{b]j}$. For an arbitrary tensor with internal SO(3)-index a, the action of ${}^{(3)}\mathcal{D}_i$ is given by,

$$ {}^{(3)}\mathcal{D}_i X^{aj} = {}^{(3)}\nabla_i X^{aj} + \omega_i{}^a{}_b X^{bj}, \tag{3.76} $$

and explicitly,

$$ {}^{(3)}\mathcal{D}_j \pi^j{}_a = \partial_j \pi^j{}_a - \omega_{ja}{}^b \pi^j{}_b, \tag{3.77} $$

since $\pi^i{}_a$ is a vector density of weight 1.

The Lagrange multiplier λ_{ab} in (3.74) enforces the primary constraint $\mathcal{J}^{ab} = 0$ associated with the kernel (3.71). $\mathcal{J}^{ab}$ is proportional to the momenta of the antisymmetric part of the vielbein velocity. This can be seen by the decomposition $E_{ai}\dot{E}^a{}_j$ into a symmetric and an antisymmetric term,

$$ E_{ai}\dot{E}^a{}_j = \tfrac{1}{2}\Big[\dot{\gamma}_{ij} + \dot{w}_{ij}\Big], \tag{3.78} $$

where $\dot{\gamma}_{ij} = 2E_{a(i}\dot{E}^a{}_{j)}$ and $\dot{w}_{ij} = 2E_{a[i}\dot{E}^a{}_{j]}$ corresponds to the velocity of the rotational degrees of freedom contained in $E^a{}_i$. The *momenta conjugate to the rotational fields* are proportional to the primary constraints $\mathcal{J}^{ab}$,

$$ \frac{\delta\mathcal{S}}{\delta\dot{w}_{ij}} = \frac{\delta\mathcal{S}}{\delta\dot{E}^a_k}\frac{\delta\dot{E}^a_k}{\delta\dot{w}_{ij}} = \tfrac{1}{2}\pi^k_a E^{al}\delta^{[i}_l\delta^{j]}_k = -\tfrac{1}{2}E^i{}_a E^j{}_b \mathcal{J}^{ab}. \tag{3.79} $$

The primary constraint $\mathcal{J}^{ab} = 0$ thus corresponds to the absence of the rotational velocities $\dot{w}_{ij}$ in the Einstein–Hilbert action, making them non-dynamical.

3.3.3 Constraint Analysis

If we introduce the non-vanishing canonical Poisson brackets,

$$ \{N(x), P(y)\} = \delta(x-y), \tag{3.80a} $$

$$ \{N^i(x), P_j(y)\} = \delta^i_j\delta(x-y), \tag{3.80b} $$

$$ \{E^a{}_i(x), \pi^j{}_b(y)\} = \delta^a_b\delta^j_i\delta(x-y), \tag{3.80c} $$

and generalise (3.39) with $\mathcal{Q}^I = (N, N^i, E^a{}_i)$ and $\Pi_I = (P, P_i, \pi^i{}_a)$ we can initiate the Dirac–Bergmann algorithm by imposing the stability of the primary constraints,

$$ \dot{P}(x) = \{P(x), H_T\} \approx \mathcal{R}(x) \approx 0, \tag{3.81a} $$

$$ \dot{P}_i(x) = \{P_i(x), H_T\} \approx \mathcal{R}_i(x) \approx 0, \tag{3.81b} $$

generating the vielbein version of the Hamiltonian and momentum constraints (3.40). The stability of the primary constraints $\mathcal{J}^{ab} \approx 0$, yields,

$$\begin{aligned}\dot{\mathcal{J}}^{ab}(x) = \{\mathcal{J}^{ab}(x), H_T\} = -\int \mathrm{d}^3y \Big[& N(y)\{\mathcal{J}^{ab}(x), \mathcal{R}(y)\} \\ & + N^i(y)\{\mathcal{J}^{ab}(x), \mathcal{R}^i(y)\} \\ & + \lambda^{cd}(y)\{\mathcal{J}^{ab}(x), \mathcal{J}_{cd}(y)\}\Big], \end{aligned} \tag{3.82}$$

where each of the brackets read,

$$\{\mathcal{J}^{ab}(x),\, \mathcal{R}(y)\,\} = 0, \tag{3.83a}$$

$$\{\mathcal{J}^{ab}(x), \mathcal{R}_i(y)\} = \mathcal{J}^{ab}(x)\frac{\partial}{\partial y^i}\delta(x-y), \tag{3.83b}$$

$$\{\mathcal{J}^{ab}(x), \mathcal{J}^{cd}(y)\} = \left[\delta^{a[c}\mathcal{J}^{d]b}(x) - \delta^{b[c}\mathcal{J}^{d]a}(x)\right]\delta(x-y), \tag{3.83c}$$

implying that $\dot{\mathcal{J}}^{ab}$ vanish trivially on the primary constraint surface $\mathcal{J}^{ab} = 0$, not generating any secondary constraints.

Imposing the stability of the secondary constraints (3.81), $\mathcal{R} \approx 0$ and $\mathcal{R}_i \approx 0$ yields,

$$\dot{\mathcal{R}}(x) \approx -\int \mathrm{d}^3y \Big[N(y)\{\mathcal{R}(x)\ , \mathcal{R}(y)\} + N^j(y)\{\mathcal{R}(x)\ , \mathcal{R}_j(y)\}\Big] \approx 0, \tag{3.84a}$$

$$\dot{\mathcal{R}}_i(x) \approx -\int \mathrm{d}^3y \Big[N(y)\{\mathcal{R}_i(x), \mathcal{R}(y)\} + N^j(y)\{\mathcal{R}_i(x), \mathcal{R}_j(y)\}\Big] \approx 0. \tag{3.84b}$$

These brackets, similar to the metric formulation (3.43), form the vielbein diffeomorphism algebra,

$$\{\mathcal{R}(x), \mathcal{R}(y)\} = \Big[\mathcal{R}^i(y)\frac{\partial}{\partial y^i} - \mathcal{R}^i(x)\frac{\partial}{\partial x^i}\Big]\delta(x-y), \tag{3.85a}$$

$$\{\mathcal{R}(x), \mathcal{R}_i(y)\} = \mathcal{R}(y)\frac{\partial}{\partial y^i}\delta(x-y), \tag{3.85b}$$

$$\{\mathcal{R}_i(x), \mathcal{R}_j(y)\} = \Big[\mathcal{R}_i(x)\frac{\partial}{\partial y^j} - \mathcal{R}_j(y)\frac{\partial}{\partial x^i}\Big]\delta(x-y). \tag{3.85c}$$

where $\mathcal{R}^i = \gamma^{ij}\mathcal{R}_j$. We emphasise that $\mathcal{R}$ and $\mathcal{R}_i$ are not the same functions as (3.36a–3.36b) and the above brackets are computed with a Poisson structure defined by different canonical brackets from (3.43).

Since (3.85) are all proportional to $\mathcal{R}$ and $\mathcal{R}_i$, they all vanish weakly, making $\dot{\mathcal{R}} \approx 0$ and $\dot{\mathcal{R}}_i \approx 0$ trivial, thus not providing any tertiary constraints. These, together with (3.83) show that all the functions $P, P_i, \mathcal{J}^{ab}, \mathcal{R}, \mathcal{R}_i$ are first class.

If we introduce the smearing function $\theta_{ab} = -\theta_{ba}$,

$$\mathcal{J}[\theta] = \int \mathrm{d}^3x\, \theta_{ab}(x)\mathcal{J}^{ab}(x), \tag{3.86}$$

$\mathcal{J}[\theta]$ acts as a generator via the Poisson bracket transforming the vielbein and its momentum,

$$\{E^a{}_i(x), \mathcal{J}[\theta^a]\} = \theta^a{}_b E^b{}_i, \tag{3.87a}$$

$$\{\pi^i{}_a(x), \mathcal{J}[\theta^a]\} = -\pi^i{}_b(x)\theta^b{}_a(x), \tag{3.87b}$$

which corresponds to an infinitesimal rotation $\Omega^a{}_b = \delta^a_b + \theta^a{}_b$.

Smearing the Hamiltonian and momentum constraints,

$$\mathcal{R}[\chi] = \int \mathrm{d}^3x\, \chi(x)\mathcal{R}(x), \tag{3.88a}$$

$$\vec{\mathcal{R}}[\xi^i] = \int \mathrm{d}^3x\, \xi^i(x)\mathcal{R}_i(x) = \int \mathrm{d}^3x\, E^a{}_i \mathcal{L}_{\vec{\xi}}\pi^i{}_a, \tag{3.88b}$$

it can be confirmed that $\mathcal{R}$ and $\mathcal{R}_i$ correspond to diffeomorphisms orthogonal to and on the spatial hypersurface, respectively. For example,

$$\{E^a{}_i, \vec{\mathcal{R}}[\xi^i]\} = -\mathcal{L}_{\vec{\xi}} E^a{}_i = -\big[\xi^j\partial_j E^a{}_i + E^a{}_j\partial_i\xi^j\big], \tag{3.89a}$$

$$\{\pi^i{}_a, \vec{\mathcal{R}}[\xi^i]\} = -\mathcal{L}_{\vec{\xi}}\pi^i{}_a = -\big[\xi^j\partial_j\pi^i{}_a - \pi^j{}_a\partial_j\xi^i + \pi^i{}_a\partial_j\xi^j\big], \tag{3.89b}$$

while,

$$\{E^a{}_i, \mathcal{R}[\chi]\} = -\chi K^j{}_i E^a{}_j, \tag{3.90}$$

where,[10]

$$K^i{}_j = \tfrac{1}{2m_{\mathrm{pl}}^2\sqrt{\gamma}}\Big[\pi^i{}_a E^a{}_j - \tfrac{1}{2}\delta^i_j\pi^k{}_a E^a{}_k\Big]. \tag{3.91}$$

In terms of the smeared generators, the constraint algebra takes the form,

$$\{\mathcal{J}[\theta_{ab}], \mathcal{J}[\theta'_{cd}]\} = \mathcal{J}[\theta^c_a\theta'_{cb} - \theta'_a{}^c\theta_{cb}], \tag{3.92a}$$

$$\{\mathcal{J}[\theta_{ab}], \vec{\mathcal{R}}[\xi^i]\} = \mathcal{J}[\xi^i\partial_i\theta_{ab}], \tag{3.92b}$$

$$\{\mathcal{J}[\theta_{ab}], \mathcal{R}[\chi]\} = 0, \tag{3.92c}$$

$$\{\vec{\mathcal{R}}[\xi^i], \vec{\mathcal{R}}[\xi'^i]\} = -\vec{\mathcal{R}}[\xi^j\partial_j\xi'^i - \xi'^j\partial_j\xi^i], \tag{3.92d}$$

$$\{\mathcal{R}[\chi], \vec{\mathcal{R}}[\xi^i]\} = \mathcal{R}[\xi^i\partial_i\chi], \tag{3.92e}$$

$$\{\mathcal{R}[\chi], \mathcal{R}[\chi']\} = \vec{\mathcal{R}}\big[\gamma^{ij}(\chi\,\partial_j\chi' - \chi'\,\partial_j\chi)\big], \tag{3.92f}$$

which is the full SO(3) *covariant diffeomorphism algebra* in terms of vielbeins.

We could now in principle construct the Castellani generators, but since $\mathcal{J}[\theta_{ab}]$ is already a primary first-class constraint and does not generate secondary constraints, it is already the full generator. The corresponding spatial and

[10] Note that this is a "generalised" extrinsic curvature which is only symmetric on the primary constraint surface $\mathcal{J}^{ab} \approx 0$.

temporal diffeomorphism generators would take the form (3.47) with $\mathcal{H} \to \mathcal{R}$ and $\mathcal{H}_i \to \mathcal{R}_i$.

Note that $\mathcal{H}_i$ and $\mathcal{R}_i$ are not the same even after a substitution $\gamma_{ij} \to E^a{}_i$ and $\pi^{ij} \to \pi^i{}_a$. In particular, $\mathcal{H}_i(E^a{}_j, \pi^k{}_b)$ only generate spatial diffeomorphisms on the symmetric combinations $(E^{\mathrm{T}}\delta E)_{ij}$ and $\pi^{(i}{}_a E^{aj)}$ leaving the rotational fields and momenta unaffected, thus not generating the expected transformation (3.89). For the algebra to take the diagonal form (3.92), it is therefore necessary to use the functions $\mathcal{R}$ and $\mathcal{R}_i$.

Hamilton's equations for the vielbein variables take the form,

$$\dot{E}^a{}_i \approx \{E^a{}_i, H_T\} \approx NK^j{}_i E^a{}_j + \mathcal{L}_{\vec{N}} E^a{}_i - \lambda^a{}_b E^b{}_i, \tag{3.93a}$$
$$\dot{\pi}^i{}_a \approx \{\pi^i{}_a, H_T\} \approx -\{\pi^i{}_a, \mathcal{R}[N]\} + \mathcal{L}_{\vec{N}} \pi^i{}_a + \lambda^b{}_a \pi^i{}_b, \tag{3.93b}$$
$$\dot{N} \approx \{N\,,\, H_T\,\} \approx -\lambda, \tag{3.93c}$$
$$\dot{N}^i \approx \{N^i, H_T\,\} \approx -\lambda^i, \tag{3.93d}$$

where,

$$\begin{aligned}\{\pi^i{}_a, \mathcal{R}[N]\} = {} & N\Big[K^i{}_j \delta^b_a - \tfrac{1}{2} E^i{}_a E^b{}_k K^k{}_j\Big]\pi^j{}_b \\ & +2Nm_{\mathrm{pl}}^2 \sqrt{\gamma}\, E_{aj}\Big[{}^{(3)}G^{ij} + \Lambda\gamma^{ij}\Big] \\ & \quad +2m_{\mathrm{pl}}^2\sqrt{\gamma}\, E_{aj}\Big[\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}\Big]{}^{(3)}\nabla_k {}^{(3)}\nabla_l N.\end{aligned} \tag{3.94}$$

With the field equations and the constraints at hand, we can now compute the dimensionality of the physical phase space. The canonical variables $(N, P), (N^i, P_i), (E^a{}_i, \pi^i{}_a)$, form a 2×13 dimensional phase space, while the 11 first-class constraints,

$$P \approx 0, \qquad P_i \approx 0, \qquad \mathcal{J}^{ab} \approx 0, \qquad \mathcal{R} \approx 0, \qquad \mathcal{R}_i \approx 0, \tag{3.95}$$

reduce the dimensionality by 2×11, resulting in a physical phase space with 2×2 degrees of freedom, corresponding to the 2 helicities of a massless spin-2 field and their momenta, in agreement with the metric formulation.

3.3.4 Restoring SO(1,3) Covariance

In the decomposition (3.65), the boost fields p^a parametrise 3 of the Lorentz fields contained in $e^A{}_\mu$, which manifestly drop out of the action in its 3+1 form (3.29). As we saw above, the resulting vielbein action retains only the SO(3) symmetry, even though the original action is fully covariant.

The full vielbein $e^A{}_\mu$ transforms as a vector under Lorentz transformations, so that infinitesimally,

$$\delta e^A{}_\mu = \theta^A{}_B e^B{}_\mu, \tag{3.96}$$

where $\theta_{AB} = -\theta_{BA}$ parametrise the transformation $L^A{}_B = \delta^A_B + \theta^A{}_B$. For $\theta_{0a} \neq 0$, such a transformation would not preserve the decomposed form of the vielbein (3.65) and the spatial metric (3.66) would not be invariant in form.

If we, however, introduce canonical momenta conjugate to the boosts,

$$P_a = \frac{\delta \mathcal{S}}{\delta \dot{p}^a}, \tag{3.97}$$

we can retain the full Lorentz symmetry on the phase-space variables.

Since the action is independent of p^a and $\dot{p}^a$, (3.97) clearly vanish and produce primary constraints $P_a \approx 0$. By generalising the rotational generator $\mathcal{J}^{ab} = \pi^{i[a} E^{b]}_i$ to also rotate p^a and its momenta P_a,

$$\mathbb{J}^{ab} = \mathcal{J}^{ab} + P^{[a} p^{b]}, \tag{3.98}$$

the smeared operator $\mathbb{J}[\vartheta_{ab}] = \int \mathrm{d}^3x\, \vartheta_{ab} \mathbb{J}^{ab}$, generates the transformations,

$$\{p^a, \mathbb{J}[\vartheta_{ab}]\} = \vartheta^a{}_b p^b, \tag{3.99a}$$

$$\{P_a, \mathbb{J}[\vartheta_{ab}]\} = -P_b \vartheta^b{}_a, \tag{3.99b}$$

yet leaves the transformations of E^a_i and $\pi^i{}_a$ unchanged (3.87). The modified generator $\mathbb{J}[\vartheta]$ transforms the components of the full vielbein e^A_μ (3.65),

$$\delta e^0{}_0 = \{\alpha N + p_b E^b{}_j N^j, \mathbb{J}[\vartheta]\} = 0, \tag{3.100a}$$

$$\delta e^0{}_i = \{p_b E^b{}_i, \mathbb{J}[\vartheta]\} = 0, \tag{3.100b}$$

$$\delta e^a{}_0 = \{N p^a + A^a{}_b E^b{}_j N^j, \mathbb{J}[\vartheta]\} = \vartheta^a{}_b e^b{}_0, \tag{3.100c}$$

$$\delta e^a{}_i = \{A^a{}_b E^b{}_i, \mathbb{J}[\vartheta]\} = \vartheta^a{}_b e^b{}_i, \tag{3.100d}$$

yielding (3.96) for pure rotations ($\theta_{a0} = 0$ and $\theta_{ab} = \vartheta_{ab}$).

From (3.96) with $\theta_{a0} = \zeta_a$ and $\theta_{ab} = 0$, the boost p^a and spatial vielbein E^a_i transform as,

$$\delta p^a = \alpha \zeta^a, \tag{3.101a}$$

$$\delta E^a_i = \frac{1}{1+\alpha}\Big[\zeta^a p_b - p^a \zeta_b\Big] E^b{}_i, \tag{3.101b}$$

implying that E^a_i transforms with a p^a and ζ^a dependent rotation. The function which generates these transformations can be shown to be [62],

$$\mathbb{K}^a = \alpha P^a + \frac{2}{1+\alpha} \mathcal{J}^{ab} p_b, \tag{3.102a}$$

$$\mathbb{K}[\zeta_a] = \int \mathrm{d}^3x \Big[\alpha \zeta_a P^a + \Theta_{ab}(\zeta, p) \mathcal{J}^{ab}\Big], \qquad \Theta_{ab} = \frac{2}{1+\alpha} \zeta_{[a} p_{b]}, \tag{3.102b}$$

where Θ_{ab} is the infinitesimal rotation resulting from two non-collinear boosts, called a Thomas–Wigner rotation.

Together, $\mathbb{J}[\vartheta_{ab}]$ and $\mathbb{K}[\zeta_a]$, now produce the infinitesimal Lorentz rotation of the vielbein,

$$\{e^A{}_\mu, \mathbb{J}[\vartheta_{ab}]\} = \theta^A{}_B e^B{}_\mu, \qquad \theta_{ab} = \vartheta_{ab}, \qquad \theta_{a0} = 0, \tag{3.103a}$$
$$\{e^A{}_\mu, \mathbb{K}[\zeta_a]\} = \theta^A{}_B e^B{}_\mu, \qquad \theta_{ab} = 0, \qquad \theta_{a0} = \zeta_a. \tag{3.103b}$$

The generators now obey the full Lorentz algebra,

$$\{\mathbb{J}[\vartheta_{ab}], \mathbb{J}[\vartheta'_{ab}]\} = \mathbb{J}\big[\vartheta_{ac}\vartheta'^c{}_b - \vartheta'_{ac}\vartheta^c{}_b\big], \tag{3.104a}$$
$$\{\mathbb{J}[\vartheta_{ab}], \mathbb{K}[\zeta_c]\} = \mathbb{K}[\vartheta_{ab}\zeta^b], \tag{3.104b}$$
$$\{\mathbb{K}[\zeta_a], \mathbb{K}[\zeta'_b]\} \approx \mathbb{J}[\zeta_a\zeta'_b - \zeta_b\zeta'_a], \tag{3.104c}$$

where the last bracket only holds weakly since it is generically not possible to construct gauge generators for the momenta off-shell [51, 63].[11] While the Hamiltonian constraint is unchanged by the extension of the phase space with (p^a, P_a), so that,

$$\{\mathbb{J}[\vartheta], \mathcal{R}[\chi]\} = 0, \qquad \{\mathbb{K}[\vec{\zeta}], \mathcal{R}[\chi]\} = 0, \tag{3.105}$$

the generator of spatial diffeomorphisms needs to be modified to also transform p^a and P_a. This generalisation takes the form,

$$\widehat{\mathcal{R}}[\vec{\xi}] = \int \mathrm{d}^3x \left[\xi^i \mathcal{R}_i + p^a \mathcal{L}_{\vec{\xi}} P_a\right], \tag{3.106}$$

which generates the transformations,

$$\{p^a, \widehat{\mathcal{R}}[\vec{\xi}]\} = -\mathcal{L}_{\vec{\xi}} p^a, \qquad \{P_a, \widehat{\mathcal{R}}[\vec{\xi}]\} = -\mathcal{L}_{\vec{\xi}} P_a, \tag{3.107}$$

on the boost variables and leaves the transformations of $E^a{}_i$ and $\pi^i{}_a$ (3.89) unchanged. It then follows that,

$$\{\mathbb{J}[\vartheta^a{}_b], \widehat{\mathcal{R}}[\xi^i]\} = \mathbb{J}[\mathcal{L}_{\vec{\xi}}\vartheta^a{}_b], \qquad \{\mathbb{K}[\zeta_a], \widehat{\mathcal{R}}[\xi^i]\} = \mathbb{K}[\mathcal{L}_{\vec{\xi}}\zeta_a]. \tag{3.108}$$

Since the Hamiltonian constraint is independent of the boost and its momenta, the modified generator still obeys,

$$\{\widehat{\mathcal{R}}[\xi^i], \widehat{\mathcal{R}}[\xi'^i]\} = -\widehat{\mathcal{R}}[\xi^j\partial_j\xi'^i - \xi'^j\partial_j\xi^i], \tag{3.109a}$$
$$\{\mathcal{R}[\chi], \widehat{\mathcal{R}}[\xi^i]\} = \mathcal{R}[\xi^i\partial_i\chi], \tag{3.109b}$$

while (3.92f) remains unchanged, which completes the full Lorentz covariant diffeomorphism algebra.

The addition of the 3 primary constraints $P_a \approx 0$ does not change the previously discussed constraint structure, as all the constraints are independent of the boosts. This implies that $\dot{P}_a \approx 0$ is trivial and that P_a weakly Poisson commutes with all the constraints, making it first-class, removing the 2×3 additional dimensions spanned by p^a and P_a.

[11] This follows from the fact that the p^a-dependent rotation Θ_{ab} only weakly transforms the momenta P_a in the correct way (i.e. $\delta_\zeta P_a = -\zeta^b P_b p_a/\alpha$).

Chapter 4

Ghosts in Multi-Gravity

In this chapter we consider the idea of multi-gravity and identify the problematic ghost-modes that generally appear in such theories. We then present an argument restricting the possibly ghost-free theories to those proposed by Hinterbichler and Rosen in 2012 [64].

4.1 Multi-Gravity

A broad class of theories can be obtained by deforming several copies of the Einstein–Hilbert action (3.9) by a non-derivative interaction term. More precisely, for a set of $\mathcal{N}$ metrics $g^I_{\mu\nu}$ or vielbeins $e^A_{I\,\mu}$, each with its own Einstein–Hilbert term $\mathcal{S}_{\mathrm{EH}}[g^I]$ and matter coupling $\mathcal{S}^I_{\mathrm{m}}[e_I,\psi_I]$, we consider actions of the form,

$$\mathcal{S} = \sum_I \Big[\mathcal{S}_{\mathrm{EH}}[g^I] + \mathcal{S}^I_{\mathrm{m}}[e_I,\psi_I]\Big] - \int \mathrm{d}^4x\, V(g_1, g_2, \ldots, g_{\mathcal{N}}), \tag{4.1}$$

where V is a non-derivative interaction potential.

Such theories produce natural extensions of General Relativity, with field equations of the form,

$$G^I_{\mu\nu} + \Lambda^I g^I_{\mu\nu} + V^I_{\mu\nu} = \frac{1}{2m_I^2} T^I_{\mu\nu}, \tag{4.2}$$

where $V^I_{\mu\nu}$ encodes the multi-metric interaction. Each matter field couples exclusively to one of the metrics, which therefore provides a natural spacetime geometry governing that matter sector. Since every metric mediates its own gravitational interaction, the resulting framework is referred to as *multi-gravity*. When the field equations are linearised around Minkowski space, the perturbations generically correspond to *spin-2 fields*, and the non-linear theory can be interpreted as a theory of *interacting spin-2 fields* [65].

The main obstruction to formulating such theories consistently is that, in addition to the spin-2 modes, they generically propagate scalar modes with *negative kinetic terms* rendering the Hamiltonian unbounded from below. These fields are called *Boulware–Deser ghosts* which, if propagating, make the theory unphysical. At the quadratic level, the relevant modes are easy to identify: as is well-known, the helicity-0 component of a metric perturbation around a fixed background carries a negative kinetic term in the linearised Einstein–Hilbert action. At the non-linear level, however, the analysis is more complicated, and has often been explained in a somewhat convoluted way, obscuring the origin of the ghost-mode. In this chapter, we present a complementary perspective that makes the ghost-mode visible at the full non-linear level and use it to identify the Boulware–Deser ghost in theories of the form (4.1). This perspective will further enable us to derive necessary conditions on non-derivative spin-2 interactions.

4.2 The Boulware–Deser Ghost

One might ask how the addition of a *non-derivative interaction* can introduce additional dynamical modes, such as the Boulware–Deser ghost, since all the kinetic terms in theories of the form (4.1) arise from the Einstein–Hilbert terms. This appears to suggest that the ghost-modes are present already in the Einstein–Hilbert terms, even though General Relativity is ghost-free. We therefore begin by analysing the Einstein–Hilbert action itself in order to identify the relevant mode.

Given the 3+1 decomposition of the metric (3.23), one can further decompose the spatial metric using the unimodular (Lichnerowicz–York) decomposition [66, 67], into a conformal scalar mode ϕ and a unimodular metric $\overline{\gamma}_{ij}$,

$$\gamma_{ij} = e^{\phi}\overline{\gamma}_{ij}, \qquad \det\overline{\gamma} = 1. \tag{4.3}$$

This induces a decomposition of the metric momenta (3.30),

$$\pi^{ij} = e^{-\phi}(\overline{\pi}^{ij} + \tfrac{1}{3}\overline{\gamma}^{ij}\pi), \qquad \overline{\gamma}_{ij}\overline{\pi}^{ij} = 0, \tag{4.4}$$

where π is conjugate to ϕ, and $\overline{\pi}^{ij}$ to $\overline{\gamma}_{ij}$. The canonical one-form factorises,

$$\pi^{ij}\dot{\gamma}_{ij} = \overline{\pi}^{ij}\dot{\overline{\gamma}}_{ij} + \pi\dot{\phi}, \tag{4.5}$$

implying that the decomposition is canonical, preserving the Poisson structure.

In terms of these new variables, the form of the Einstein–Hilbert Hamiltonian (3.35) is unchanged,

$$H_T = -\int \mathrm{d}^3x \Big[N\mathcal{H} + N^i\mathcal{H}_i + \lambda P + \lambda^i P_i\Big], \tag{4.6}$$

while the Hamiltonian and momentum constraints (3.36a–3.36b) takes the form,[1]

[1]The spatial curvature ${}^{(3)}R$ can be written in terms of the curvature of $\overline{\gamma}_{ij}$, and spatial derivatives of the scalar mode, ${}^{(3)}R = {}^{(3)}\overline{R} - 2\overline{\nabla}^2\phi - \frac{1}{2}\overline{\gamma}^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j\phi$. Similarly $\sqrt{\gamma} = e^{3\phi/2}$.

$$\mathcal{H} = m_{\rm pl}^2\sqrt{\gamma}\Big[{}^{(3)}R - 2\Lambda\Big] + \frac{1}{m_{\rm pl}^2\sqrt{\gamma}}\Big[\tfrac{1}{6}\pi^2 - \overline{\pi}_{ij}\overline{\pi}^{ij}\Big], \tag{4.7a}$$

$$\mathcal{H}_i = 2\nabla_k\big(\overline{\gamma}_{ij}\overline{\pi}^{kj}\big) + \tfrac{2}{3}\nabla_i(\pi). \tag{4.7b}$$

The terms quadratic in the momenta in π^2 and $\overline{\pi}^{ij}\overline{\pi}_{ij}$ appear with *different sign* in the total Hamiltonian so that,

$$H_T \propto -\tfrac{1}{6}\pi^2 + \overline{\pi}^{ij}\overline{\pi}_{ij}, \tag{4.8}$$

implying that (ϕ, π) is a *ghost* since $N > 0$ by construction.

In General Relativity, these modes are not propagating since the Hamiltonian constraint $\mathcal{H} \approx 0$ eliminates π in terms of $\phi, \overline{\gamma}_{ij}$ and $\overline{\pi}^{ij}$,

$$\pi^2 \approx 6\Big[\overline{\pi}^{ij}\overline{\pi}_{ij} - m_{\rm pl}^4 e^{3\phi}\big({}^{(3)}R - 2\Lambda\big)\Big], \tag{4.9}$$

while Hamilton's equation for ϕ determines $\dot{\phi}$ in terms of $\pi(\overline{\pi}^{ij}, \overline{\gamma}_{ij})$, so the *ghost is only present off shell* and is thus not propagating, rendering General Relativity ghost-free.

However, if we now add a non-derivative potential $\mathcal{V}(g)$ to the action, the ghost will generically be propagating. To see this, we note that the addition of a potential does not affect the canonical momenta and only produces an additional term $\mathcal{V}(N, N^i, \phi, \overline{\gamma})$ in the Hamiltonian,

$$H_T = -\int \mathrm{d}^3x\Big[N\mathcal{H} + N^i\mathcal{H}_i + \lambda P + \lambda^i P_i - \mathcal{V}\Big]. \tag{4.10}$$

The stability of the lapse momentum $\dot{P} \approx 0$ yields the modified Hamiltonian constraint,

$$\{P, H_T\} = \mathcal{H}(\phi, \pi, \overline{\gamma}, \overline{\pi}) - \frac{\partial\mathcal{V}}{\partial N}(N, N^i, \phi, \overline{\gamma}) \approx 0, \tag{4.11}$$

which generically determines the lapse instead of the ghost-mode, or if solved for ϕ, promotes the lapse variable to a dynamical field which generically is ghostly. Unless there are further constraints which eliminate the ghost-modes, the theory becomes inconsistent.

4.3 Ghost-isolating Ansatz

The identification of the ghostly mode which appears in the Einstein–Hilbert term is useful in the context of theories of the form (4.1), which include well-studied examples such as Massive Gravity and Bimetric Theory. The non-derivative interaction does not affect the kinetic terms or the canonical momenta, implying that the analogous scalar mode will appear in the Hamiltonian with the wrong sign, rendering it a ghost. It is therefore essential that the theory has enough constraints to render these modes non-dynamical or the theory will be inconsistent and must be discarded.

For a given non-trivial potential, the full non-linear constraint analysis is already technically demanding for $\mathcal{N} = 1$ and $\mathcal{N} = 2$, see e.g. [68, 69]. A perturbative expansion can verify the absence of the ghost order by order, but establishing ghost-freedom for generic field configurations requires a non-linear analysis. The unimodular decomposition and the identification of the conformal mode ϕ as the ghost, however, suggest a simplifying Ansatz that retains enough structure to derive necessary conditions on the non-derivative interaction.

By eliminating the spatial dependence and the unimodular mode $\overline{\gamma}_{ij}$, one retains only the lapse and the conformal mode of the spatial metric. We introduce the *ghost-isolating Ansatz*,[2]

$$g_{\mu\nu}(t) = \begin{pmatrix} -N^2(t) & 0 \\ 0 & f^2(\phi(t))\delta_{ij} \end{pmatrix}, \tag{4.12}$$

where f is an appropriate function of the scalar mode ϕ. Substituting the Ansatz with $f(\phi) = \left(\frac{3}{16}\right)^{1/3}\phi^{2/3}$ into (4.1) for each metric yields,

$$\mathcal{S} = -\int \mathrm{d}t \left[\sum_{I=1}^{\mathcal{N}} \frac{\dot{\phi}_I^2}{2N_I} + \mathcal{V}(N_I, \phi_I) \right], \tag{4.13}$$

where, since $N_I > 0$, the kinetic terms for the modes ϕ_I carry the wrong sign. These are precisely the *Boulware–Deser ghosts* that generically appear in non-linear theories of massive spin-2 fields.

If we introduce momenta conjugate to ϕ_I and N_I,

$$\pi_I(t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_I(t)} = -\frac{\dot{\phi}_I(t)}{N_I(t)}, \tag{4.14a}$$

$$P_I(t) = \frac{\partial \mathcal{L}}{\partial \dot{N}_I(t)} = 0, \tag{4.14b}$$

[2] One could in principle consider a more general vielbein Ansatz including Lorentz fields, but for locally Lorentz covariant theories the Lorentz degrees of freedom are trivial in the sense that their equations of motion always admit trivial solutions.

the absence of time derivatives of the lapse yields the primary constraints $\mathcal{C}^I_{(1)} = P_I \approx 0$. Inverting the momentum-velocity relation, $\dot{\phi}_I = -N_I\pi_I$, the total Hamiltonian takes the form,

$$H_T = H_C + \sum_I \lambda_I P^I, \qquad\qquad H_C = -\sum_I N_I \mathcal{R}^I + \mathcal{V}, \tag{4.15}$$

for Lagrange multipliers λ_I enforcing the primary constraints $P_I \approx 0$, and we have defined,

$$\mathcal{R}^I(t) = \tfrac{1}{2}\pi_I^2(t). \tag{4.16}$$

Since the momenta enter the Hamiltonian as $H_T \propto -N_I\pi_I^2$, all the modes (ϕ_I, π_I) are ghostly and must be non-propagating if the theory is to be ghost-free. While this reduced system is considerably simpler than the full canonical analysis described in the previous chapter, it retains enough of the constraint structure to yield *necessary conditions* on the interaction potential for ghost-freedom. For the theory to be consistent, the constraint structure must render all ϕ_I and their momenta non-propagating, leaving a trivial physical phase space with *no remaining degrees of freedom*.

We now proceed with the Dirac–Bergmann algorithm to determine whether this reduced parametrisation has sufficient constraints to eliminate the ghost-modes ϕ_I and their momenta π_I. Imposing the stability of the primary constraints $\mathcal{C}^I_{(1)} = P^I \approx 0$ yields the conditions,

$$\dot{\mathcal{C}}^I_{(1)} = \{\mathcal{C}^I_{(1)}, H_T\} = \mathcal{R}^I - \mathcal{V}^I \approx 0, \qquad\qquad \mathcal{V}^I = \frac{\partial \mathcal{V}}{\partial N_I}. \tag{4.17}$$

These are explicitly independent of the Lagrange multipliers λ_I and non-vanishing, and therefore constitute secondary constraints,

$$\mathcal{C}^I_{(2)} = \mathcal{R}^I - \mathcal{V}^I \approx 0. \tag{4.18}$$

Their stability must in turn be imposed,

$$\dot{\mathcal{C}}^I_{(2)} = \{\mathcal{C}^I_{(2)}, H_C\} + \sum_J \mathcal{V}^{IJ}\lambda_J \approx 0, \qquad\qquad \mathcal{V}^{IJ} = \frac{\partial^2 \mathcal{V}}{\partial N_I \partial N_J}. \tag{4.19}$$

The *number of tertiary constraints* is determined by how many Lagrange multipliers λ_J remain undetermined by (4.19), which is given by the *co-rank of the Hessian* $\mathcal{V}^{IJ}$.

From this result, we can directly *rule out the case of full-rank* $\mathcal{V}^{IJ}$: if the Hessian has full rank, all Lagrange multipliers λ_I are determined by (4.19) and no tertiary constraints arise. Moreover, the bracket between the primary and secondary constraints,

$$\{\mathcal{C}^I_{(1)}, \mathcal{C}^J_{(2)}\} = \mathcal{V}^{IJ}, \tag{4.20}$$

is again full rank, rendering *all constraints second class*.

The initial phase space, spanned by (ϕ_I, π_I) and (N_I, P^I), is $4\mathcal{N}$ dimensional, and the $2\mathcal{N}$ second-class constraints reduce it via (2.38) to a $2\mathcal{N}$-dimensional physical phase space, corresponding precisely to $\mathcal{N}$ dynamical Boulware–Deser ghosts. A potential whose Hessian has full rank is therefore *not ghost-free* and must be considered inconsistent.

If, on the other hand, the potential is linear in all the lapses so that $\mathcal{V}^{IJ} = 0$, the secondary constraints $\mathcal{C}^I_{(2)} \approx 0$ can no longer determine the lapses and instead eliminate the ghost momenta π_I. Since (4.19) is then independent of the Lagrange multipliers, the conditions $\dot{\mathcal{C}}^I_{(2)} \approx 0$ yield *tertiary constraints* $\mathcal{C}^I_{(3)} = \{\mathcal{C}^I_{(2)}, H_C\} \approx 0$, which eliminate ϕ_I. Their stability in turn produces *quaternary constraints* $\mathcal{C}^I_{(4)} = \dot{\mathcal{C}}^I_{(3)} \approx 0$ which determine the lapses, so that all phase-space variables are determined, leaving a trivial physical phase space and *no propagating ghosts*.

More generally, by parametrising the rank of $\mathcal{V}^{IJ}$ and performing a systematic constraint analysis at each rank, one can argue that the only consistent case is precisely that of vanishing Hessian [70]. A potential *linear in the lapses N_I* is therefore *necessary for the theory to be ghost-free*.

4.4 Hinterbichler–Rosen Potential

Reintroducing the spatial dependence and the remaining fields, potentials that are linear in the lapses are most easily written in terms of vielbeins e_I, since their first column $e^A_{I\,0}$ is linear in the lapse (3.65),

$$e^A{}_0 = \begin{pmatrix} \alpha N + p_b E^b{}_j N^j \\ N p^a + A^a{}_b E^b{}_j N^j \end{pmatrix}, \tag{4.21}$$

in contrast to the metric which is quadratic $g_{00} \propto -N^2$, (3.23). If we also restrict to an interaction potential which is locally Lorentz covariant and a scalar density under general coordinate transformations, the most general such potential can be written as the components of a top-form,

$$\mathcal{V}\mathrm{d}^4x = \mathcal{V}_{\alpha\beta\gamma\delta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta, \tag{4.22}$$

where $\mathcal{V}_{\alpha\beta\gamma\delta}(e_I)$ is some generic totally antisymmetric tensor contraction of the vielbeins $e^A_{I\,\mu}$. Using the 1-form expression of the vielbeins, $e^A_{I\,\mu}\mathrm{d}x^\mu = e^A_I$, the contraction can be expressed as an antisymmetric product of vielbeins,

$$\mathcal{V}\mathrm{d}^4x = \sum_{IJKL=1}^{\mathcal{N}} \mathcal{V}^{IJKL}_{ABCD}\, e^A_I \wedge e^B_J \wedge e^C_K \wedge e^D_L. \tag{4.23}$$

Since the wedge product is totally antisymmetric in the Lorentz indices, the coefficient $\mathcal{V}^{IJKL}_{ABCD}(e)$ must be proportional to ϵ_{ABCD}, as all other basis tensors of rank-4 Lorentz invariants are symmetric in at least one pair of indices and therefore vanish upon contraction with the wedge product. The requirement that the potential be at most linear in the lapses then forces the remaining scalar coefficient to be constant, since any non-trivial Lorentz scalar built from the vielbeins would necessarily introduce additional lapse dependence [70].

One thus obtains the *most general vielbein potential* which is a locally Lorentz invariant scalar density and at most linear in the lapses,

$$\mathcal{V} = \sum_{IJKL=1}^{\mathcal{N}} \beta^{IJKL}\, \delta^{\alpha\beta\gamma\delta}_{ABCD}\, e^{A}_{I\,\alpha} e^{B}_{J\,\beta} e^{C}_{K\,\gamma} e^{D}_{L\,\delta}, \tag{4.24}$$

where $\delta^{\alpha\beta\gamma\delta}_{ABCD} = \epsilon_{ABCD}\,\epsilon^{\alpha\beta\gamma\delta}$ and β^{IJKL} is a totally symmetric constant parameter.

This is precisely the interaction potential conjectured by Hinterbichler and Rosen [64] to be the most general ghost-free non-derivative multi-gravity interaction. Here, however, this form has been derived as a necessary consequence of ghost-freedom, rather than postulated. In the next chapter, we will see that this class of potentials contains all known ghost-free interactions, but is not ghost-free for arbitrary β^{IJKL}.

Chapter 5

On the Uniqueness of Multi-Gravity

In this chapter we derive necessary conditions on the interaction parameters β^{IJKL} of the Hinterbichler–Rosen potential (4.24) for the theory to be ghost-free. We first show that, while the potential is linear in the lapses by construction, the Lorentz constraints generically reintroduce lapse-dependent secondary constraints, destroying the ghost-eliminating structure. Using a simplifying Ansatz, we then derive conditions on β^{IJKL} under which this obstruction is avoided. For irreducible interactions with $\mathcal{N} \geq 3$, these conditions single out the Hassan–Schmidt–May determinant coupling as the unique consistent interaction, while for $\mathcal{N} = 2$, Bimetric Theory is recovered. We conclude by discussing reducible interactions and the uniqueness of ghost-free non-derivative multi-gravity theories.

5.1 Lapse-independent SO(3) Constraints

The Hinterbichler–Rosen theory is *not ghost-free* for arbitrary β^{IJKL} [71–73]. In contrast to the analysis in Chapter 4, the obstruction does not arise from an a priori non-linearity of the potential in the lapses. Instead, the issue lies in the structure of the constraints and how their solutions *reintroduce lapse dependence*.

In the covariant formulation, the Lorentz fields appear only in the potential, since the Einstein–Hilbert terms can be expressed purely in terms of the metrics. Consequently, variation of the action with respect to the Lorentz fields produces algebraic equations determined entirely by the potential. More concretely, since the Einstein–Hilbert variation (3.10a) yields terms proportional to the Einstein tensor or the metric, which are symmetric, the antisymmetric part of the field equations contains only the non-derivative potential. The antisymmetric contraction [28, 64],

$$\frac{\delta \mathcal{S}}{\delta e^{[C}_{I\,\alpha}}\eta_{D]E}e^{E}_{I\,\alpha} = 0, \tag{5.1}$$

therefore yields algebraic equations, called *Lorentz constraints*, which correspond to the variation of the action with respect to the Lorentz fields contained in the vielbeins. For the Hinterbichler–Rosen potential, contracting (5.1) with ϵ^{ABCD} yields the Lorentz constraints in the useful form,

$$\begin{aligned}\mathcal{C}^{AB}_{I} &= \epsilon^{ABCD}\frac{\delta V}{\delta e^{C}_{I\,\alpha}}\eta_{DE}e^{E}_{I\,\alpha} \\ &= \sum_{JKL}\epsilon^{\alpha\beta\gamma\delta}\beta^{IJKL}(e^{\mathrm{T}}_{I}\eta e_{J})_{\alpha\beta}e^{[A}_{K\,\gamma}e^{B]}_{L\,\delta} = 0.\end{aligned} \tag{5.2}$$

To see the obstruction to the constraint structure it is enough to consider the shift-less Ansatz,

$$e^{A}_{I\,\mu} = \begin{pmatrix} N_I & 0 \\ 0 & E^{a}_{I\,i}\end{pmatrix}, \tag{5.3}$$

and further decompose $E^{a}_{I\,i} = \Omega^{a}_{I\,b}\overline{E}^{b}_{I\,i}$, where $\Omega_I \in \mathrm{SO}(3)$ and $\overline{E}^{a}_{I\,i}$ depends only on the spatial metric degrees of freedom. The Lorentz constraints (5.2) then reduce to,

$$\mathcal{C}^{a}_{I} = \sum_{JKL}N_J\beta^{IJKL}\epsilon^{ikl}(E^{\mathrm{T}}_{I}\delta E_K)_{ik}E^{a}_{L\,l} = 0. \tag{5.4}$$

This constitutes $3(\mathcal{N}-1)$ equations for the rotational fields $\Omega^{a}_{I\,b}$, modulo the overall SO(3) frame. Since these equations are linear in the lapses, the solutions for the rotations generically acquire lapse dependence, $\Omega_I = \Omega_I(N)$. Substituting these solutions back into the potential renders it non-linear in the lapses. As a consequence, the analogues of the secondary constraints $\mathcal{C}^{I}_{(2)} \approx 0$ (4.18) become lapse-dependent, and the analogue of the Hessian $\mathcal{V}^{IJ}$ in (4.19) is no longer zero. The constraints $\mathcal{C}^{I}_{(2)} \approx 0$ then determine the lapses, while $\dot{\mathcal{C}}^{I}_{(2)} \approx 0$ fix the associated Lagrange multipliers, and the *ghost-eliminating constraints are lost.*[1] It is therefore necessary that the Lorentz constraints admit solutions for the rotations which are lapse-independent. This is, however, not true for generic β^{IJKL} and $\mathcal{N} > 2$.

[1] Due to time-reparametrisation symmetry one lapse remains undetermined, but the resulting constraints only suffice to eliminate a single ghost-mode and its momenta.

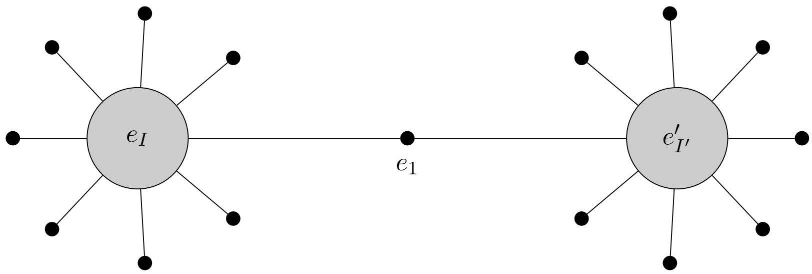

Figure 5.1: In this interaction graph, each • corresponds to a vielbein, and a line between them an abstract interaction term. A single vielbein e_1 couples directly to two otherwise disconnected interacting sectors.

5.1.1 Irreducible Multi-Gravity Interactions

Before proceeding, we introduce the notion of an *irreducible multi-gravity interaction*:

Irreducible interaction ($\mathcal{N} \geq 3$): For every pair $I \neq J$ there exist K, L such that $\beta^{IJKL} \neq 0$ with at least one of K, L distinct from both I and J.

This restricts the interactions to genuine multi-field couplings, where at least three distinct fields interact directly, as opposed to purely pairwise interactions. It also excludes *reducible* configurations in which otherwise genuinely multi-field sectors are connected only through a single shared field, as illustrated in Figure 5.1.

We will now show that the only irreducible interactions which admit lapse-independent solutions for the rotations are given by a fully factorised form of the interaction parameters,

$$\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L. \tag{5.5}$$

Any totally symmetric tensor β^{IJKL} admits a *symmetric outer product decomposition* [74], i.e. there exists R pairwise non-proportional vectors β_r^I and constants c_r, so that,

$$\beta^{IJKL} = \sum_{r=1}^{R} c_r \, \beta_r^I \, \beta_r^J \, \beta_r^K \, \beta_r^L, \tag{5.6}$$

where R is called the symmetric rank. Using this decomposition, the Lorentz constraints (5.4) factorise as,

$$\mathcal{C}_I^a = \sum_r \sum_J N_J \beta_r^J \, t_{rI}^a = 0, \tag{5.7}$$

where t^a_{rI} are functions only of the spatial vielbeins,

$$t^a_{rI} = c_r \beta^I_r \epsilon^{ijk} \left[E^T_I \delta U_r\right]_{ij} U^a_{r\,k}, \qquad U^a_{r\,i} = \sum_I \beta^I_r E^a_{I\,i}. \tag{5.8}$$

The $\mathcal{C}^a_I = 0$ are linear in N_J, so for the system to admit lapse-independent solutions each coefficient must vanish independently,

$$X^a_{IJ} = \sum_r \beta^J_r \, t^a_{rI} = 0, \tag{5.9}$$

for all I, J and a. While the Lorentz constraints $\mathcal{C}^a_I = 0$ constitute $3(\mathcal{N}-1)$ equations for $3(\mathcal{N}-1)$ rotational variables, the conditions (5.9) yield $3\mathcal{N}(\mathcal{N}-1)$ equations for the same unknowns, and are generically overdetermined. A consistent solution for generic vielbeins therefore requires an $\mathcal{N}$-fold redundancy in (5.9).

Since the decomposition vectors β^I_r are only required to be pairwise non-proportional, they may be linearly dependent, and their number R can exceed the dimension s of their linear span. It is therefore convenient to introduce a basis b^I_n ($n = 1, \ldots, s$) of this span, so that $\beta^I_r = \sum_n A^n{}_r \, b^I_n$ for some coefficients $A^n{}_r$. For fixed I and a, the conditions (5.9) can then be written as the matrix equation $\mathbb{B}\,\Theta^a_I = 0$, where $\mathbb{B}$ is the $\mathcal{N} \times s$ matrix formed by the basis vectors and Θ^a_I collects the corresponding linear combinations $\theta^a_{nI} = \sum_r A^n{}_r \, t^a_{rI}$,

$$\mathbb{B} = \begin{pmatrix} b^1_1 & \cdots & b^1_s \\ \vdots & & \vdots \\ b^{\mathcal{N}}_1 & \cdots & b^{\mathcal{N}}_s \end{pmatrix}, \qquad \Theta^a_I = \begin{pmatrix} \theta^a_{1I} \\ \vdots \\ \theta^a_{sI} \end{pmatrix}. \tag{5.10}$$

Since the columns of $\mathbb{B}$ are linearly independent by construction, the matrix has rank s and trivial right kernel, forcing $\Theta^a_I = 0$ and hence $\theta^a_{nI} = 0$ for all n. For $s \geq 2$, these constitute s systems of equations on the $3(\mathcal{N}-1)$ rotational variables, which generically overconstrain the system. The detailed argument, presented in Paper I, shows that for irreducible interactions with $R \geq 2$, the individual conditions $\theta^a_{nI} = 0$ cannot all be simultaneously satisfied for generic vielbeins, and no cancellation between different t^a_{rI} can produce the required redundancy when $s > 1$.

The only remaining possibility is $s = 1$, meaning all decomposition vectors are proportional, $\beta^I_r = \lambda_r \beta^I$, so that,

$$\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L, \tag{5.11}$$

up to an overall constant. If we insert this into the Hinterbichler–Rosen potential, it reduces to the determinant of a linear combination of the vielbeins,

$$V_{\text{det}} = \frac{2m^4}{4!} \sum_{IJKL} \beta^I \beta^J \beta^K \beta^L \delta^{\alpha\beta\gamma\delta}_{ABCD} e^A_{I\,\alpha} e^B_{J\,\beta} e^C_{K\,\gamma} e^D_{L\,\delta} \tag{5.12}$$

$$= 2m^4 \det\left(\sum_I \beta^I e_I\right). \tag{5.13}$$

This is precisely the Hassan–Schmidt–May interaction first presented in [75], and the above argument establishes it as the *unique irreducible multi-gravity coupling* within the Hinterbichler–Rosen class that admits lapse-independent Lorentz constraints.

5.1.2 Bimetric Theory

Before proceeding with the analysis of the Hassan–Schmidt–May theory in full, we note that the above proof does not cover every possible multi-gravity interaction, since we have restricted to irreducible interactions with $\mathcal{N} \geq 3$. For $\mathcal{N} = 2$, the Lorentz constraints (5.4) always admit lapse-independent solutions, since one can always solve them by imposing the symmetrisation condition,

$$[E_1^{\mathrm{T}} \delta\, E_2]_{[ij]} = 0, \tag{5.14}$$

which places no restriction on β^{IJKL}. In the fully covariant theory, the Lorentz constraints (5.2) are similarly solved by,

$$[e_1^{\mathrm{T}} \eta\, e_2]_{[\mu\nu]} = 0. \tag{5.15}$$

This condition implies that the matrix $e_1^{-1} e_2$ can be expressed purely in terms of the metrics [41, 76],

$$(e_1^{-1} e_2)^2 = g_1^{-1} g_2 \qquad \Longrightarrow \qquad S = e_1^{-1} e_2 = \sqrt{g_1^{-1} g_2}\,. \tag{5.16}$$

Substituting this into the Hinterbichler–Rosen potential (4.24) for $\mathcal{N} = 2$, the sum over $IJKL$ reduces to [64],

$$V_{\mathrm{bi}} = 2m^4 \sqrt{-g_1} \sum_{n=0}^{4} \frac{\alpha_n}{n!(4-n)!}\, \delta^{\mu_1 \ldots \mu_n \mu_{n+1} \ldots \mu_4}_{\nu_1 \ldots \nu_n \nu_{n+1} \ldots \nu_4}\, S^{\nu_1}{}_{\mu_1} \ldots S^{\nu_n}{}_{\mu_n}, \tag{5.17}$$

where $S^{\mu}{}_{\nu} = (e_1^{-1} e_2)^{\mu}{}_{\nu}$ and α_n are defined by the β^{IJKL}, so that,

$$\alpha_n = \beta^{\overbrace{1 \ldots 1}^{4-n}\overbrace{2 \ldots 2}^{n}}. \tag{5.18}$$

The contraction of $S^{\nu}{}_{\mu}$ with the generalised delta can be expressed in terms of the *elementary symmetric polynomials*,

$$\begin{aligned} &\mathcal{P}_0(S) = 1, \qquad \mathcal{P}_1(S) = [S], \qquad \mathcal{P}_2(S) = \tfrac{1}{2}\left([S]^2 - [S^2]\right) \\ &\mathcal{P}_3(S) = \tfrac{1}{6}\left([S]^3 - 3[S][S^2] + 2[S^3]\right), \qquad \mathcal{P}_4(S) = \det(S)\,. \end{aligned} \tag{5.19}$$

so that the potential reads,

$$V_{\mathrm{bi}} = 2m^4 \sqrt{-g_1} \sum_{n=0}^{4} \alpha_n \mathcal{P}_n(S). \tag{5.20}$$

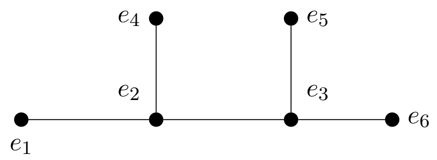

Figure 5.2: A bimetric tree interaction on six vielbeins. Each • represents a vielbein and each edge a bimetric interaction $V_{\rm bi}(e_I, e_J)$. The leaves e_1, e_4, e_5, e_6 each participate in a single interaction.

This is precisely the ghost-free bimetric potential of Hassan and Rosen [36, 40], confirming that the Hinterbichler–Rosen class for $\mathcal{N} = 2$ reproduces Bimetric Theory with arbitrary coupling constants, in agreement with the known ghost-freedom proof [41, 42, 77].

5.2 Reducible Interactions

The restriction to irreducible interactions excludes a number of known ghost-free theories, which can be constructed by combining the two consistent building blocks: the Bimetric $V_{\rm bi}(e_I, e_J)$ and the Hassan–Schmidt–May $V_{\rm det}$ potentials. These *reducible* interactions have β^{IJKL} of symmetric rank $R \geq 2$, yet can still admit lapse-independent Lorentz constraints. The key requirement is that the interaction graph has a *tree structure*.

We now introduce the notion of an interaction graph, $\mathcal{G}$. $\mathcal{G}$ is a *bipartite* graph with two different types of vertices:

– Vielbein vertices • for each vielbein e_I.

– Interaction vertices ∘ for either a determinant or bimetric interaction.

Every vielbein which interacts with either a determinant or a bimetric potential is connected to ∘ by an edge –, •–∘. For two vielbeins interacting via a bimetric interaction, we can suppress ∘ and write •–• as short for •–∘–•. A vielbein in $\mathcal{G}$ which couples only to one ∘ or • is called a *leaf*.

Pairwise bimetric interactions of the form,

$$V = \sum_{(IJ)\in\mathcal{G}} V_{\rm bi}(e_I, e_J), \tag{5.21}$$

where $\mathcal{G}$ is a tree graph on $\mathcal{N}$ vertices, i.e. $(I, J) \in \mathcal{G}$ if there is a bimetric interaction between the Ith and Jth vielbein, see for example Figure 5.2. Such interactions were first proposed in [78] and have been shown to be ghost-free [79, 80]. The mechanism is that the Lorentz constraints can be solved iteratively from the leaves of the tree inward: each leaf vielbein $e_{I'}$ is symmetrised with its unique neighbour $e_{J'}$, i.e. $[e_{I'}^{\mathrm{T}}\eta e_J]_{[\mu\nu]} = 0$, and once these rotations are fixed, the tree is essentially reduced to a new tree, with new leaves $e_{J'}$. This iterative procedure then determines all but one overall Lorentz frame by pairwise symmetrising each of the vielbeins in the tree.

The β-parameters for pairwise Bimetric Theory are of the form,

$$\beta^{IJKL} = \sum_{(I'J')\in\mathcal{G}} \beta^{IJKL}_{(I'J')}, \tag{5.22}$$

where each $\beta^{IJKL}_{(I'J')}$ is only non-zero when $IJKL$ are either I' or J' which are connected in the tree $\mathcal{G}$ and are related to the Bimetric parameters $\alpha_n^{(I'J')}$ via a generalisation of (5.18).

The same leaf-removal procedure applies to sums of determinant couplings,

$$V_{\mathrm{det}} = 2m^4 \sum_r \det\Big(\sum_{I\in\mathcal{A}_r} \beta_r^I e_I\Big), \tag{5.23}$$

where the index sets $\mathcal{A}_r$ share at most one vielbein and the interaction graph forms a tree. If we then consider a vielbein $e_{I'}$ which appears in exactly one sector, i.e. a leaf, say the one with index set $\mathcal{A}_r$, its Lorentz constraints reduce to,

$$[e_{I'}^{\mathrm{T}}\eta u_r]_{[\mu\nu]} = 0, \qquad u_r = \sum_{I\in\mathcal{A}_r} \beta_r^I e_I. \tag{5.24}$$

Each sector's Lorentz constraints therefore decouple, and the shared vielbein's equation is satisfied once the rotations within each sector have been determined. If no such leaf vielbein exists, i.e. if every vielbein in a given sector also appears in at least one other, the tree can first be reduced by the leaf-removal procedure described above for Bimetric Theory.

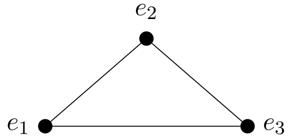

Figure 5.3: A bimetric cycle on three vielbeins. Every vielbein participates in two bimetric interactions and there are no leaves.

If the interaction graph contains a *cycle*, the tree mechanism fails. For example, three pairwise bimetric interactions forming a cycle,

$$V_{\mathrm{bi}}(e_1, e_2) + V_{\mathrm{bi}}(e_2, e_3) + V_{\mathrm{bi}}(e_3, e_1)$$

, have no leaves, see Figure 5.3, and solving the Lorentz constraints for one vielbein generically fixes its rotation through a combined condition involving multiple neighbours, without setting any individual symmetrisation to zero. The remaining equations then overconstrain the system, and lapse-independent solutions do not exist for generic vielbeins. Such cycle interactions have been shown to be inconsistent both in the metric formulation [72, 81] and in the vielbein formulation [71, 73], though the two are not equivalent since the vielbeins are not pairwise symmetrised.

More generally, any combination of Bimetric and determinant building blocks can be assembled, provided the resulting interaction graph remains a tree, see Figure 5.4 for an example. This includes mixed configurations where some edges correspond to pairwise Bimetric couplings and some vertices represent determinant sectors sharing single vielbeins. The most general multi-gravity theory within the Hinterbichler–Rosen class (4.24) that admits lapse-independent Lorentz constraints is therefore a *tree of Bimetric and determinant interactions*, encompassing all currently known ghost-free multi-gravity theories. A detailed analysis and explicit examples are presented in Paper I [70].

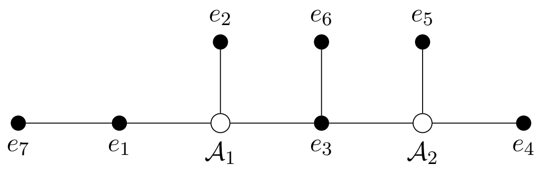

Figure 5.4: A mixed interaction tree on seven vielbeins. The determinant sectors $\mathcal{A}_1 = \{e_1, e_2, e_3\}$ and $\mathcal{A}_2 = \{e_3, e_4, e_5\}$ share the vielbein e_3, which also interacts with e_6 via a bimetric edge. A second bimetric edge connects e_1 to e_7.

5.3 Uniqueness of Ghost-free Multi-Gravity

The results of the preceding sections, together with the analysis of Chapter 4, lead to a strong restriction on the space of consistent non-derivative multi-gravity theories of the form (4.1). The chain of necessary conditions can be summarised as follows:

i) The ghost-isolating Ansatz and the constraint analysis of Section 4.2 showed that *ghost-freedom requires* the interaction potential to be *linear in all the lapses* [70].

ii) Imposing local Lorentz covariance and general coordinate invariance on such a lapse-linear potential *uniquely singles out the Hinterbichler–Rosen form* (4.24), parametrised by the totally symmetric tensor β^{IJKL}.

iii) The analysis of the Lorentz constraints showed that for ghost-free theories, the rotational constraints must admit *lapse-independent solutions*, which is not true for generic β^{IJKL}.

iv) For irreducible interactions with $\mathcal{N} \geq 3$, the only form of β^{IJKL} which admits lapse-independent solutions for the rotations is $\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L$, corresponding to the *Hassan–Schmidt–May determinant coupling*, while for $\mathcal{N} = 2$ the parameters are unconstrained and reproduce Bimetric Theory.

This leaves only reducible interactions. If the theory is reducible, there exists by definition at least one pair of vielbeins e_I and e_J which do not interact directly, meaning $\beta^{IJKL} = 0$ for all K, L. This implies that e_I and e_J belong to separate interaction sectors which communicate at most indirectly through shared vielbeins. Each such sector, viewed in isolation, must itself admit lapse-independent Lorentz constraints, since the rotational equations for vielbeins appearing in only one sector are determined entirely within that sector. By the results above, each sector with three or more directly interacting vielbeins must be a Hassan–Schmidt–May determinant coupling, while sectors involving only two vielbeins correspond to bimetric interactions.

The most general reducible multi-gravity theory within the Hinterbichler–Rosen class that admits lapse-independent Lorentz constraints is therefore a tree of irreducible building blocks, bimetric interactions and Hassan–Schmidt–May determinant couplings, joined through single shared vielbeins. We *conjecture* that these tree-structured combinations, which encompass all currently known ghost-free multi-gravity theories, constitute the most *general ghost-free non-derivative interaction* between multiple spin-2 fields with Einstein–Hilbert kinetic terms. In the next chapter we proceed to argue that the Hassan–Schmidt–May theory is in fact ghost-free.

Chapter 6

Ghost-free Multi-Gravity

As we saw in the last chapter, the only irreducible multi-gravity theories with the correct constraint structure, and which are potentially ghost-free, are given by either Bimetric or Hassan–Schmidt–May theory. Bimetric Theory has been shown to be ghost-free [34–36, 40–42, 77], and in this chapter we present the results of Paper III, arguing that the Hassan–Schmidt–May theory is ghost-free as well, by performing a Hamiltonian constraint analysis and computing the dimension of the physical phase space.

6.1 Covariant Ghost-free Multi-Gravity

The Hassan–Schmidt–May action reads,

$$\mathcal{S} = \int \mathrm{d}^4 x \left[\sum_{I=1}^{\mathcal{N}} m_I^2 \sqrt{-g_I} (R_I - 2\Lambda_I) - V_{\mathrm{det}} \right] + \sum_{I=1}^{\mathcal{N}} \mathcal{S}_{\mathrm{m}}^I [e_I, \psi_I], \tag{6.1}$$

where m_I, R_I and Λ_I are Planck-mass-like parameters, Ricci scalars and cosmological constants for each of the vielbeins e_I. The potential can be written in terms of the determinant of a sum of vielbeins,

$$V_{\mathrm{det}} = 2m^4 \det(u), \qquad u^A{}_\mu = \sum_I \beta^I e_I{}^A{}_\mu. \tag{6.2}$$

Each vielbein sector can couple to matter fields ψ_I via $\mathcal{S}_{\mathrm{m}}$, but each matter field can only interact with one vielbein, and exclusively with other fields in that sector. While each Einstein–Hilbert term is separately invariant under local Lorentz transformations and diffeomorphisms, the interaction potential preserves only the diagonal subgroup, i.e. the transformations for which all vielbeins transform simultaneously with the same Lorentz and diffeomorphism parameters.

Every vielbein has a corresponding covariant derivative ${}^I\nabla_\mu$ and associated

Christoffel symbols,

$$ {}^{I}\nabla_\mu g^I_{\alpha\beta} = 0, \tag{6.3} $$

$$ {}^{I}\Gamma^\mu{}_{\alpha\beta} = \tfrac{1}{2} g_I^{\mu\nu}\Big[\partial_\alpha g^I_{\nu\beta} + \partial_\beta g^I_{\alpha\nu} - \partial_\nu g^I_{\alpha\beta}\Big], \tag{6.4} $$

from which the Ricci curvatures R_I are constructed via (3.6) and (3.7). We can also introduce the locally Lorentz covariant derivatives ${}^{I}\mathcal{D}_\mu$, compatible with the vielbein $e_I{}^A{}_\mu$,

$$ {}^{I}\mathcal{D}_\mu e_I{}^A{}_\nu = {}^{I}\nabla_\mu e_I{}^A{}_\nu + {}^{I}\omega_\mu{}^A{}_B e_I{}^B{}_\nu = 0, \tag{6.5} $$

where ${}^{I}\nabla_\mu$ is blind to the internal Lorentz indices and ${}^{I}\omega_{\mu AB}$ is the SO(1,3) spin connection with components,

$$ {}^{I}\omega_{\mu AB} = e^\alpha_{I[A}\, {}^{I}\nabla_\mu e^I_{B]\alpha}. \tag{6.6} $$

Variation of the action using the analogues of (3.10a) and (3.10b) yields the covariant multi-gravity field equations, which take the form of deformed Einstein equations for each vielbein $e_I{}^A{}_\mu$,

$$ G^I_{\mu\nu} + \Lambda_I g^I_{\mu\nu} + V^I_{\mu\nu} = \frac{1}{2m_I^2} T^I_{\mu\nu}, \tag{6.7} $$

where $G^I_{\mu\nu} = R^I_{\mu\nu} - \frac{1}{2}R^I g^I_{\mu\nu}$ is the Einstein tensor constructed from the curvatures of $g^I_{\mu\nu}$ (3.6–3.7), and the vielbein interaction term $V^I_{\mu\nu}$ reads,

$$ V^I_{\mu\nu} = \beta^I \frac{m^4}{3!\, m_I^2\sqrt{-g_I}} \delta^{\alpha\beta\gamma\delta}_{ABCD} e_I{}^A{}_\mu u^B{}_\beta u^C{}_\gamma u^D{}_\delta g^I_{\alpha\nu}, \tag{6.8} $$

and the stress-energy tensor is defined as usual,[1]

$$ T^I_{\mu\nu} = -\frac{1}{\sqrt{-g_I}} \frac{\delta \mathcal{S}_\mathrm{m}}{\delta e_I{}^\mu{}_A} \eta_{AB} e_I{}^B{}_\nu. \tag{6.9} $$

The Lorentz constraints $V^I_{[\mu\nu]} = 0$ reduce to the symmetrisation conditions,

$$ \beta^I e_I^\mathrm{T} \eta\, u = \beta^I u^\mathrm{T} \eta\, e_I, \tag{6.10} $$

for every I. These are not all independent, as summing over I yields $u^\mathrm{T}\eta\, u = (u^\mathrm{T}\eta\, u)^\mathrm{T}$, which is trivially satisfied, resulting in at most $6(\mathcal{N}-1)$ independent constraints, enough to determine the non-dynamical Lorentz fields after an overall Lorentz frame is fixed.

[1] For Dirac fields, this has to be interpreted as the Belinfante–Rosenfeld tensor.

6.2 Canonical Ghost-free Multi-Gravity

We now proceed to reformulate Hassan–Schmidt–May multi-gravity theory in terms of the Hamiltonian formulation and reproduce the results in Paper III.

Each of the vielbeins $e^A_{I\,\mu}$ can be decomposed as (3.65),

$$e^A_{I\,\mu} = \begin{pmatrix} \alpha_I & p^I_b \\ p^a_I & A^a_{I\,b} \end{pmatrix} \begin{pmatrix} N_I & 0 \\ E^b_{I\,j} N^j_I & E^b_{I\,i} \end{pmatrix}, \tag{6.11}$$

where $\alpha_I = \sqrt{1 + p^a_I \delta_{ab} p^b_I}$ and $A^a_{I\,b} = \delta^a_b + p^a_I p^I_b/(1+\alpha_I)$, while the Einstein–Hilbert terms in (6.1) can be reformulated in the same way as in Section 3.3.[2] The conjugate momenta are defined analogously,

$$\pi^i_{I\,a} = \frac{\delta \mathcal{S}}{\delta \dot{E}^a_{I\,i}}, \qquad P^I = \frac{\delta \mathcal{S}}{\delta \dot{N}_I}, \qquad P^I_i = \frac{\delta \mathcal{S}}{\delta \dot{N}^i_I}, \tag{6.12}$$

resulting in the primary constraints,

$$\mathcal{J}^{ab}_I = \pi^{i[a}_I E^{b]}_{I\,i} \approx 0, \qquad P^I \approx 0, \qquad P^I_i \approx 0. \tag{6.13}$$

In contrast to the Einstein–Hilbert action, the Lorentz fields do not drop out of the interaction potential (6.2) and explicitly depend on the boosts p^a_I and the rotational fields. While the momenta of the rotations are contained in $\pi^i_{I\,a}$, we define the momenta conjugate to the boosts separately,

$$P^I_a = \frac{\delta \mathcal{S}}{\delta \dot{p}^a_I}, \tag{6.14}$$

and since they only appear in the non-derivative interaction, we get the additional primary constraints,

$$P^I_a \approx 0. \tag{6.15}$$

If we introduce the canonical Poisson structure with the non-vanishing brackets,

$$\{E^a_{I\,i}(x), \pi^j_{J\,b}(y)\} = \delta_{IJ}\delta^a_b \delta^j_i \delta(x-y), \tag{6.16a}$$

$$\{N_I(x), P^J(y)\} = \delta_{IJ}\delta(x-y), \tag{6.16b}$$

$$\{N^i_I(x), P^J_j(y)\} = \delta_{IJ}\delta^i_j \delta(x-y), \tag{6.16c}$$

$$\{p^a_I(x), P^J_b(y)\} = \delta_{IJ}\delta^a_b \delta(x-y), \tag{6.16d}$$

we can compute the Poisson bracket of arbitrary phase-space functions using (3.39) with the generalised coordinates $\mathcal{Q}^I = (N_I, N^i_I, E^a_{I\,i}, p^a_I)$ and $\Pi_I = (P^I, P^I_i, P^I_a, \pi^i_{I\,a})$.

[2] This assumes that there exists a hypersurface which is spatial with respect to their corresponding metrics, which we will assume throughout.

Since only the first column of the vielbeins contains the lapses and shifts,

$$e^A{}_0 = \begin{pmatrix} \alpha N + p_b E^b{}_j N^j \\ N p^a + A^a{}_b E^b{}_j N^j \end{pmatrix}, \tag{6.17}$$

and the determinant of a matrix is linear in the columns, the potential decomposes into,

$$V_{\text{det}} = -\sum_{I=1}^{\mathcal{N}} \Big[N_I \widetilde{\mathcal{C}}^I + N^i_I \widetilde{\mathcal{C}}^I_i \Big], \tag{6.18a}$$

$$\widetilde{\mathcal{C}}^I = -2m^4 \beta_I \det(U) \Big[\alpha_I - \sum_{J=1}^{\mathcal{N}} \beta_J p^J_a E^a_{J\,i} U^i_b p^b_I \Big], \tag{6.18b}$$

$$\widetilde{\mathcal{C}}^I_i = -2m^4 \beta_I \det(U) \Big[p^I_a - \sum_{J=1}^{\mathcal{N}} \beta_J p^J_c E^c_{J\,j} U^j_b A^b_{I\,a} \Big] E^a_{I\,i}, \tag{6.18c}$$

where we have introduced the sum of the spatial vielbeins $U^a_i = \sum_I \beta_I e^a_{I\,i}$, or in terms of $E^a_{I\,i}$ and $A^a_{I\,b}$,

$$U^a_i = \sum_{I=1}^{\mathcal{N}} \beta_I A^a_{I\,b} E^b_{I\,i}. \tag{6.19}$$

The non-derivative interaction leaves the form of the Einstein–Hilbert terms (3.74) intact. Since both the Einstein–Hilbert terms and the interaction potential are linear in the lapses and shifts, the total Hamiltonian,

$$H_T = -\int \mathrm{d}^3x \sum_I \Big[N_I \mathcal{C}^I_{(2)} + N^i_I \mathcal{C}^I_i + \lambda_I P^I + \lambda^i_I P^I_i + \lambda^a_I P^I_a + \lambda^I_{ab} \mathcal{J}^{ab}_I \Big], \tag{6.20}$$

is linear in N_I and N^i_I. The functions $\mathcal{C}^I_{(2)} = \mathcal{R}^I + \widetilde{\mathcal{C}}^I$ and $\mathcal{C}^I_i = \mathcal{R}^I_i + \widetilde{\mathcal{C}}^I_i$ are independent of lapses and shifts, and $\mathcal{R}^I$ and $\mathcal{R}^I_i$ are the direct generalisations of (3.75a) and (3.75b),

$$\mathcal{R}^I = m^2_I \sqrt{\gamma_I} \Big[{}^{(3)}R_I - 2\Lambda_I \Big] + \tfrac{1}{4m^2_I \sqrt{\gamma_I}} \Big[\tfrac{1}{2} (\pi^i_{I\,a} E^a_{I\,i})^2 - \pi^i_{I\,a} E^a_{I\,j} \pi^j_{I\,b} E^b_{I\,i} \Big], \tag{6.21a}$$

$$\mathcal{R}^I_i = E^a_{I\,i}\, {}^I\!\mathcal{D}_j \pi^j_{I\,a} + {}^I\omega_{iab} \mathcal{J}^{ab}_I, \tag{6.21b}$$

where ${}^I\omega_{iab} = E^k_{I[a}\, {}^I\nabla_i E^I_{b]k}$ is the spatial spin connection.

6.3 Constraint Analysis

With the total Hamiltonian (6.20) and the primary constraints (6.13–6.14) at hand, the Dirac–Bergmann algorithm starts by imposing the stability of the primary constraints,

$$\dot{P}^I = \{P^I, H_T\} \approx \mathcal{C}^I_{(2)} \approx 0, \tag{6.22a}$$

$$\dot{P}^I_i = \{P^I_i, H_T\} \approx \mathcal{C}^I_i \approx 0, \tag{6.22b}$$

$$\dot{P}^I_a = \{P^I_a, H_T\} \approx 0, \tag{6.22c}$$

$$\dot{\mathcal{J}}^{ab}_I = \{\mathcal{J}^{ab}_I, H_T\} \approx 0, \tag{6.22d}$$

where $\dot{P}^I_a \approx 0$ and $\dot{\mathcal{J}}^{ab}_I \approx 0$ correspond to the Lorentz constraints, which can more conveniently be written in their covariant form,

$$\mathcal{C}^I_{\mu\nu} = \sum_J \beta_J [e^{\mathrm{T}}_I \eta e_J]_{[\mu\nu]} \approx 0, \tag{6.23}$$

which in 3+1 variables take the form,

$$\begin{aligned} \mathcal{C}^I_{i0} = \tfrac{1}{2} \sum_J \beta_J H^{IJ}_{ab} \Big[&\tfrac{1}{\alpha_J} E^a_{I\,i}\, p^b_J N_J - \tfrac{1}{\alpha_I} E^a_{J\,i}\, p^b_I N_I \\ &+ E^a_{I\,i}\, E^b_{J\,j}\, N^j_J - E^b_{J\,i}\, E^a_{I\,j}\, N^j_I \Big] \approx 0, \end{aligned} \tag{6.24}$$

$$\mathcal{C}^I_{ij} = \sum_J \beta_J H^{IJ}_{ab} E^a_{I\,[i} E^b_{J\,j]} \approx 0, \tag{6.25}$$

where $H^{IJ}_{ab} = A^c_{I\,a} \delta_{cd} A^d_{J\,b} - p^I_a p^J_b$.

As we saw in Chapter 5, it is crucial that the Lorentz constraints admit lapse-independent solutions for the rotations. Since (6.25) is independent of lapses and shifts, this is indeed the case here. Consequently, the secondary constraints $\mathcal{C}^I_{(2)} \approx 0$, $\mathcal{C}^I_i \approx 0$ or $\mathcal{C}^I_{ij} \approx 0$ carry no explicit dependence on N_I. If we then determine the boosts p^a_I using the constraints $\mathcal{C}^I_i \approx 0$, the solutions depend only on $E^a_{I\,i}$ and $\pi^i_{I\,a}$, so that,

$$p^a_I = p^a_I(E, \pi). \tag{6.26}$$

The spatial Lorentz constraints $\mathcal{C}^I_{ij} \approx 0$ can then be solved for the rotational components of $E^a_{I\,i}$ in terms of only $E^a_{I\,i}$, $\pi^i_{I\,a}$ and $p^a_I(E, \pi)$, so that all Lorentz fields are determined.

This leaves the remaining secondary constraints $\mathcal{C}^I_{(2)} \approx 0$ as functions of $E^a_{I\,i}$ and $\pi^i_{I\,a}$ and, crucially, independent of the lapses. This means that $\mathcal{C}^I_{(2)} \approx 0$ constrains the dynamical variables and that their stability yields tertiary constraints.

Because $\sum_I \beta_I \mathcal{C}^I_{\mu\nu} = 0$, one can only determine $\mathcal{N} - 1$ copies of the shift vectors and rotational fields, leaving one set undetermined. Similarly, it can easily be verified that the sum $\sum_I \widetilde{\mathcal{C}}^I_i$ vanishes trivially, leaving the combination $\sum_I \mathcal{C}^I_i$ independent of boosts, so that one boost vector remains undetermined. The residual boost and rotations are a manifestation of the local Lorentz invariance, and can be determined by choosing a gauge of the internal Lorentz space. The residual shift is fixed by choosing a gauge for the spatial diffeomorphism.

Since $\mathcal{C}^I_{\mu\nu} \approx 0$ and $\mathcal{C}^I_i \approx 0$ were solved for non-dynamical variables (i.e. variables whose momenta are determined by primary constraints), imposing their stability, $\dot{\mathcal{C}}^I_{\mu\nu} \approx 0$ and $\dot{\mathcal{C}}^I_i \approx 0$, determines Lagrange multipliers $\lambda^I_{ab}, \lambda^a_I, \lambda^I_i$ and produces no further constraints. However, the stability of $\mathcal{C}^I_{(2)} \approx 0$ yields the conditions,

$$\begin{aligned}\{\mathcal{C}^I_{(2)}(x), H_T\} = -\int \mathrm{d}^3y \sum_J \Big[& N_J(y)\{\mathcal{C}^I_{(2)}(x), \mathcal{C}^J_{(2)}(y)\} \\ & + N^i_J(y)\{\mathcal{C}^I_{(2)}(x), \mathcal{C}^J_i(y)\} \\ & + \lambda^a_J(y)\{\mathcal{C}^I_{(2)}(x), P^J_a(y)\} \\ & + \lambda^J_{ab}(y)\{\mathcal{C}^I_{(2)}(x), \mathcal{J}^{ab}_J(y)\}\Big] \approx 0. \end{aligned} \tag{6.27}$$

While these constraints depend on the Lagrange multipliers λ^a_I and λ^I_{ab}, these have already been determined, so $\dot{\mathcal{C}}^I_{(2)} \approx 0$ constrains the remaining canonical variables $E^a_{I\,i}$ and $\pi^i_{I\,a}$. However, were these conditions to determine the lapses, the theory would not have enough constraints to eliminate the $\mathcal{N}-1$ ghosts. Therefore, it is crucial that $\dot{\mathcal{C}}^I_{(2)} \approx 0$ cannot consistently determine N_I.

6.4 Equal Boost Ansatz

That this is the case can be shown under a simplifying Ansatz, where all boost fields p^a_I are weakly set to be equal. The constraints $\mathcal{C}^I_{i0} \approx 0$ can then be explicitly solved, effectively setting all the shifts to be equal, and the stability conditions (6.27) weakly reduce to the form,

$$\dot{\mathcal{C}}^I_{(2)} \approx \beta_I \sum_J \mathcal{M}_{IJ} N_J \widetilde{\mathcal{C}}^J \approx 0, \tag{6.28}$$

where the matrix $\mathcal{M}$ is at most rank-2 with only $\mathcal{N}-1$ independent components,

$$\mathcal{M}_{IJ} = \left[\frac{\delta \mathcal{R}^I}{\delta \pi^i_{I\,a}} - \frac{\delta \mathcal{R}^J}{\delta \pi^i_{J\,a}}\right]\left[\Big(\sum_K \beta_K E_K\Big)^{-1}\right]^i_{\;a} = X_I - X_J, \tag{6.29a}$$

$$X_I = \frac{\delta \mathcal{R}^I}{\delta \pi^i_{I\,a}}\left[\Big(\sum_K \beta_K E_K\Big)^{-1}\right]^i_{\;a}. \tag{6.29b}$$

For $\mathcal{N} > 2$ the tertiary constraints (6.28) can thus determine at most 2 of the lapses, and the remaining equations would constrain $E^a_{I\,i}$ and $\pi^i_{I\,a}$. However, if we solve (6.28) explicitly for, say $N_{\mathcal{N}-1}$ and $N_{\mathcal{N}}$,[3]

$$N_{\mathcal{N}-1} = \frac{1}{\beta_{\mathcal{N}}} \frac{\sum_{I=1}^{\mathcal{N}-2} [X_{\mathcal{N}} - X_I] N_I \beta_I}{X_{\mathcal{N}-1} - X_{\mathcal{N}}}, \tag{6.30a}$$

$$N_{\mathcal{N}} = -\frac{1}{\beta_{\mathcal{N}-1}} \frac{\sum_{I=1}^{\mathcal{N}-2} [X_{\mathcal{N}-1} - X_I] N_I \beta_I}{X_{\mathcal{N}-1} - X_{\mathcal{N}}}. \tag{6.30b}$$

the solution is inconsistent for two reasons. First, $N_{\mathcal{N}}$ and $N_{\mathcal{N}-1}$ cannot both be strictly positive, contradicting the condition $N_I > 0$ for all I. Second, the solution (6.30) makes the potential V (6.18a) vanish, making the interaction trivially zero, and invalidating the previous constraints and assumption that $u^A_{\ \mu} = \sum_I \beta^I e^A_{I\,\mu}$ is invertible.

So the only physical branch is given by $\mathcal{M}_{IJ} \approx 0$ for all I and J. One might be worried that this corresponds to $\mathcal{N}(\mathcal{N}-1)/2$ conditions, but due to the structure $\mathcal{M}_{IJ} \propto X_I - X_J$, it is enough to impose the $\mathcal{N}-1$ tertiary constraints,

$$\mathcal{C}^I_{(3)} = \mathcal{M}_{I1} \approx 0 \qquad\qquad \Longrightarrow \qquad\qquad \mathcal{M}_{IJ} \approx 0. \tag{6.31}$$

Using the unimodular decomposition introduced in Section 4.2, it can be shown explicitly that $\mathcal{C}^I_{(2)} \approx 0$ and $\mathcal{C}^I_{(3)} \approx 0$ determine the ghost and its momenta respectively.

The stability of the tertiary constraint $\mathcal{C}^I_{(3)} \approx 0$ yields $\mathcal{N}-1$ quaternary constraints,

$$\mathcal{C}^I_{(4)}(x) = \dot{\mathcal{C}}^I_{(3)}(x) \approx -\int \mathrm{d}^3 y \sum_J N_J(y) \{\mathcal{C}^I_{(3)}(x), \mathcal{C}^J_{(2)}(y)\} \approx 0, \tag{6.32}$$

where $\{\mathcal{C}^I_{(3)}(x), \mathcal{C}^J_{(2)}(y)\}$ corresponds to the components of a $(\mathcal{N}-1)\times\mathcal{N}$ matrix, so that (6.32) can be solved for $\mathcal{N}-1$ of the N_I in terms of one residual lapse, which is determined by gauge fixing corresponding to temporal diffeomorphisms.

Since $\mathcal{C}^I_{(4)} \approx 0$ are solved for non-dynamical variables, their stability determines the Lagrange multipliers λ_I, implying that we have found all the constraints.

6.5 Hamilton's Equations

We can now evaluate Hamilton's equations for the canonical variables. The equations for the spatial vielbeins reproduce the definition of the vielbein momenta,

$$\dot{E}^a_{I\,i} \approx \{E^a_{I\,i}, H_T\} \approx N_I K^j_{I\,i} E^a_{I\,j} + \mathcal{L}_{\vec{N}_I} E^a_{I\,i} - \lambda^a_{I\,b} E^b_{I\,i}, \tag{6.33}$$

[3] Note that $\widetilde{\mathcal{C}}^J = -2m^4 \beta^J \det\left(\sum_I \beta^I E_I\right)$ so that the non-vanishing determinant factors cancel.

but with the lapse N_I, shift N_I^i and Lagrange multiplier $\lambda^a_{I\,b}$ determined by the constraints in terms of the canonical fields, and where,

$$K^i_{I\,j} = \frac{1}{2m_I^2\sqrt{\gamma_I}}\Big[\pi^i_{I\,a}\,E^a_{I\,j} - \tfrac{1}{2}\delta^i_j\pi^k_{I\,a}E^a_{I\,k}\Big]. \tag{6.34}$$

The equations for the vielbein momenta $\pi^i_{I\,a}$ contain the dynamical content,

$$\begin{aligned}\dot{\pi}^i_{I\,a} \approx \{\pi^i_{I\,a}, H_T\} \approx &- \{\pi^i_{I\,a}, \mathcal{R}^I[N_I]\} - \{\pi^i_{I\,a}, V\}\\ &+ \mathcal{L}_{\vec{N}_I}\pi^i_{I\,a} + \lambda^b_{I\,a}\pi^i_{I\,b},\end{aligned} \tag{6.35}$$

where $\{\pi^i_{I\,a}, \mathcal{R}^I[N_I]\}$ takes the same form as in General Relativity (3.94),

$$\begin{aligned}\{\pi^i_{I\,a}, \mathcal{R}^I[N_I]\} &= N_I\Big[K^i_{I\,j}\delta^b_a - \tfrac{1}{2}E^i_{I\,a}E^b_{I\,k}K^k_{I\,j}\Big]\pi^j_{I\,b}\\ &+2N_I m_I^2\sqrt{\gamma_I}\,E^I_{aj}\Big[{}^{(3)}G^{ij}_I + \Lambda_I\gamma^{ij}_I\Big]\\ &\quad+2m_I^2\sqrt{\gamma_I}\,E^I_{aj}\Big[\gamma^{ij}_I\gamma^{kl}_I - \gamma^{ik}_I\gamma^{jl}_I\Big]{}^I\nabla_k{}^I\nabla_l N_I,\end{aligned} \tag{6.36}$$

and the interaction yields,

$$\{\pi^i_{I\,a}, V\} = \frac{\partial V}{\partial E^i_{I\,a}} = -\sum_{J=1}^{\mathcal{N}}\left[N_J\frac{\partial\widetilde{\mathcal{C}}^J}{\partial E^a_{I\,i}} + N^j_J\frac{\partial\widetilde{\mathcal{C}}^J_j}{\partial E^a_{I\,i}}\right]. \tag{6.37}$$

The dynamics of the lapses, shifts and boosts are determined entirely in terms of the Lagrange multipliers which were fixed by the Dirac–Bergmann algorithm,

$$\dot{N}_I \approx \{N_I, H_T\} \approx -\lambda_I(\pi, E), \tag{6.38a}$$
$$\dot{N}^i_I \approx \{N^i_I, H_T\} \approx -\lambda^i_I(\pi, E), \tag{6.38b}$$
$$\dot{p}^a_I \approx \{\,p^a_I\,, H_T\} \approx -\lambda^a_I(\pi, E). \tag{6.38c}$$

We emphasise that for all but one species I, these Lagrange multipliers have been determined in terms of the canonical fields, while one residual copy of $\lambda, \lambda^i_I, \lambda^a_I$ and λ^I_{ab} are left undetermined until a local Lorentz frame and a gauge for the remaining lapse and shift have been chosen.

6.6 Symmetry Generators and Constraint Classification

Before computing the dimension of the physical phase space we will classify the constraints. Following the procedure in Section 3.3.4, the diagonal rotation and boost generators can easily be constructed,

$$\mathbb{J}[\vartheta^a{}_b] = \int \mathrm{d}^3x \sum_I \vartheta_{ab} \Big[\mathcal{J}_I^{ab} + P_I^{[a} p_I^{b]} \Big], \tag{6.39a}$$

$$\mathbb{K}[\zeta_a] = \int \mathrm{d}^3x \sum_I \Big[\alpha_I \zeta_a P_I^a + \Theta^I_{ab}(\zeta, p_I) \mathcal{J}_I^{ab} \Big], \tag{6.39b}$$

where the Thomas–Wigner rotations are parametrised by,

$$\Theta^I_{ab} = \frac{2}{1+\alpha_I} \zeta_{[a} p^I_{b]}. \tag{6.40}$$

These form the diagonal Lorentz algebra identical to (3.104),

$$\{\mathbb{J}[\vartheta_{ab}], \mathbb{J}[\vartheta'_{ab}]\} = \mathbb{J}\big[\vartheta_{ac}\vartheta'^c{}_b - \vartheta'_{ac}\vartheta^c{}_b\big], \tag{6.41a}$$

$$\{\mathbb{J}[\vartheta_{ab}], \mathbb{K}[\zeta_c]\} = \mathbb{K}[\vartheta_{ab}\zeta^b], \tag{6.41b}$$

$$\{\mathbb{K}[\zeta_a], \mathbb{K}[\zeta'_b]\} \approx \mathbb{J}[\zeta_a\zeta'_b - \zeta_b\zeta'_a]. \tag{6.41c}$$

Since all the other constraints are Lorentz scalars, $\mathbb{J}[\vartheta^a{}_b]$ and $\mathbb{K}[\zeta^a]$ have vanishing brackets with them, making these combinations first class, while the remaining combinations $\mathcal{J}_I^{ab}$ and P_a^I are second-class.

Using that $\sum_I \widetilde{\mathcal{C}}_i^I$ vanishes identically, it is straightforward to see from (3.88b) that,

$$\int \mathrm{d}^3x \, \xi^i(x) \sum_I \mathcal{C}_i^I(x) = \int \mathrm{d}^3x \sum_I E^a_{I\,i} \mathcal{L}_{\vec{\xi}} \pi^i_{I\,a}. \tag{6.42}$$

If we then, by analogy with Section 3.3.4, add the terms generating spatial diffeomorphisms for the boosts and their momenta, we get the generator of diagonal spatial diffeomorphisms,[4]

$$\vec{D}[\xi^i] = \int \mathrm{d}^3x \sum_I \Big[E^a_{I\,i} \mathcal{L}_{\vec{\xi}} \pi^i_{I\,a} + p_I^a \mathcal{L}_{\vec{\xi}} P_a^I \Big], \tag{6.43}$$

with the corresponding algebra,

$$\{\vec{D}[\xi^i], \vec{D}[\xi'^i]\} = -\vec{D}[\xi^j\partial_j\xi'^i - \xi'^j\partial_j\xi^i]. \tag{6.44}$$

Since all the constraints are scalar densities, their bracket with $\vec{D}[\xi^i]$ yields spatial derivatives of the constraints, which vanish weakly, making $\vec{D}[\xi^i]$ first class. It also follows that the corresponding sum of the shift momenta $\sum_I P_i^I$ is first class, as all constraints but $\mathcal{C}_{i0}^I$ are shift-independent, and,

$$\sum_I \{P_i^I, \mathcal{C}_{j0}^J\} = \mathcal{C}_{ij}^J \approx 0, \tag{6.45}$$

[4] This could in principle be generalised to the full Castellani generator (3.47) to also transform the lapse and shift momenta, but this is not needed here.

vanish weakly.

The generator of temporal diffeomorphisms is not obtained explicitly, as this needs explicit solutions of the tertiary and quaternary constraints. Yet, since the theory is fully diffeomorphism invariant, we are guaranteed to have one first-class secondary constraint associated with deformations of the spatial hypersurface. The associated primary constraint will also be first class, while the remaining combinations $\mathcal{C}^I_{(2)}$ are all second class.

With all the constraints obtained and classified (see Table 6.1 for a summary), the dimensionality of the physical phase space can be computed. The initial phase space corresponds to $\mathcal{N}$ copies of the variables $N_I, N^i_I, E^a_{I\,i}, p^a_I$ and their momenta, constituting a $2 \times 16\mathcal{N}$ dimensional phase space. The generators corresponding to temporal and spatial diffeomorphisms, and the Lorentz transformations, yield 10 first-class constraints. The combinations of momenta that produce the Hamiltonian and momentum constraints are also first class, yielding another 4, while the remaining constraints are all second class. Using (2.38), we then compute,

$$2{\times}16\,\mathcal{N} - 2{\times}14 - 22\,(\mathcal{N}-1) = 2{\times}\big[2 + 5\,(\mathcal{N}-1)\big], \tag{6.46}$$

which is the correct number of degrees of freedom for one massless and $\mathcal{N}-1$ massive spin-2 fields, confirming that the theory does not propagate the Boulware–Deser ghost-modes. In the next chapter, we identify the spin-2 structure explicitly and analyse the mass spectrum.

Constraints	Total Number	Number of 1^{st} Class	Number of 2^{nd} Class
Primary			
$P^I = 0$	$\mathcal{N}$	1	$\mathcal{N}-1$
$P^I_i = 0$	$3\mathcal{N}$	3	$3(\mathcal{N}-1)$
$P^I_a = 0$	$3\mathcal{N}$	3	$3(\mathcal{N}-1)$
$\mathcal{J}^{ab}_I = 0$	$3\mathcal{N}$	3	$3(\mathcal{N}-1)$
Secondary			
$\mathcal{C}^I_{(2)} \approx 0$	$\mathcal{N}$	1	$\mathcal{N}-1$
$\mathcal{C}^I_i \approx 0$	$3\mathcal{N}$	3	$3(\mathcal{N}-1)$
$\mathcal{C}^I_{\mu\nu} \approx 0$	$6(\mathcal{N}-1)$	0	$6(\mathcal{N}-1)$
Tertiary			
$\mathcal{C}^I_{(3)} \approx 0$	$\mathcal{N}-1$	0	$\mathcal{N}-1$
$\dot{\mathcal{C}}^I_i \approx 0$	0	0	0
$\dot{\mathcal{C}}^I_{\mu\nu} \approx 0$	0	0	0
Quaternary			
$\mathcal{C}^I_{(4)} \approx 0$	$\mathcal{N}-1$	0	$\mathcal{N}-1$
		14	$22(\mathcal{N}-1)$

Table 6.1: Summary of the constraints obtained from the Hamiltonian constraint analysis of Hassan–Schmidt–May theory and their classification into first and second class.

Chapter 7

Perturbative Expansion and Mass Spectrum

This chapter reviews the standard metric perturbative expansion of the Einstein–Hilbert action and then introduces the covariant vielbein expansion developed in Paper IV, in which the vielbein is expanded in terms of independent metric and Lorentz perturbations. We next study proportional-background solutions of the Hassan–Schmidt–May multi-gravity theory and show that the associated consistency conditions always admit solutions with positive parameters. Finally, we expand the multi-gravity interaction potential to quadratic order, diagonalise the resulting action, identify the mass eigenfields, and discuss the resulting mass spectrum.

7.1 Perturbative Einstein–Hilbert

Einstein's field equations are notoriously complicated to solve exactly, and only a handful of exact solutions are known. It can therefore be useful to linearise the field equations, or to perturbatively expand the action. A generic metric can be expanded around a background metric $\overline{g}_{\mu\nu}$ as,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{7.1}$$

for a perturbation $h_{\mu\nu}$. The background metric $\overline{g}_{\mu\nu}$ is often chosen to satisfy some field equations, such as Einstein's equations (3.12).

Using (7.1), the Einstein–Hilbert Lagrangian can be expanded to quadratic order,

$$\mathcal{L}^{(0)}_{\rm EH} = m^2_{\rm pl}\sqrt{-g}\Big[\overline{R} - 2\Lambda\Big], \tag{7.2a}$$

$$\mathcal{L}^{(1)}_{\rm EH} = -m^2_{\rm pl}\sqrt{-g}\Big[\overline{G}_{\alpha\beta} + \Lambda\overline{g}_{\alpha\beta}\Big]h^{\alpha\beta}, \tag{7.2b}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm EH} = m^2_{\rm pl}\sqrt{-g}\Big[&\tfrac{1}{2}h^{\mu\nu}\Big(\mathcal{E}_{\mu\nu}{}^{\alpha\beta} - \tfrac{1}{2}\big(\overline{g}_{\mu\nu}\overline{R}^{\alpha\beta} - (\overline{R}-2\Lambda)\delta^\alpha_\mu\delta^\beta_\nu\big)\Big)h_{\alpha\beta} \\ &+ \Big(\overline{G}_{\alpha\beta} + \Lambda\overline{g}_{\alpha\beta}\Big)\Big(h^\alpha{}_\sigma h^{\sigma\beta} - \tfrac{1}{4}hh^{\alpha\beta}\Big)\Big],\end{aligned} \tag{7.2c}$$

where $h = h^\mu{}_\mu$, $\overline{\nabla}_\mu$ is the covariant derivative compatible with the background metric $\overline{g}\mu\nu$, and the kinetic term is given by the Lichnerowicz operator,

$$\begin{aligned}\mathcal{E}_{\mu\nu}{}^{\alpha\beta} = \tfrac{1}{2}\Big[&\delta^\alpha_\mu\delta^\beta_\nu\overline{\nabla}^2 - \overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\overline{\nabla}^2 + \overline{g}^{\alpha\beta}\overline{\nabla}_\mu\overline{\nabla}_\nu \\ &+ \overline{g}_{\mu\nu}\overline{\nabla}^\alpha\overline{\nabla}^\beta - 2\delta^\alpha_{(\mu}\overline{\nabla}^\beta\overline{\nabla}_{\nu)}\Big].\end{aligned} \tag{7.3}$$

The quadratic action with Lagrangian (7.2c) is invariant under the transformation,

$$h_{\mu\nu} \mapsto h_{\mu\nu} + 2\overline{\nabla}_{(\mu}\xi_{\nu)}, \tag{7.4}$$

which corresponds to linearised diffeomorphisms.

Imposing the background Einstein equations $\overline{G}_{\alpha\beta} + \Lambda\overline{g}_{\alpha\beta} = 0$, and coupling $h_{\mu\nu}$ to the stress-energy, the linearised field equations take the form,

$$\Big[\mathcal{E}_{\mu\nu}{}^{\alpha\beta} + \Lambda\big(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\big)\Big]h_{\alpha\beta} = \frac{1}{2m^2_{\rm pl}}T_{\mu\nu}. \tag{7.5}$$

The structure of the field equations, together with linearised diffeomorphism invariance, implies that only 2 of the 10 components of $h_{\mu\nu}$ are dynamical. In Minkowski space, $\overline{g}_{\mu\nu} = \eta_{\mu\nu}$, the perturbation $h_{\mu\nu}$ transforms as the tensor representation of a massless spin-2 field, so that General Relativity may be regarded as a non-linear theory of a massless spin-2 field.

At the linear level, one can add a mass term to (7.5),

$$\begin{aligned}\Big[&\mathcal{E}_{\mu\nu}{}^{\alpha\beta} + \Lambda\big(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\big) \\ &- \tfrac{m^2_{\rm FP}}{2}\big(\delta^\alpha_\mu\delta^\beta_\nu - \overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\big)\Big]h_{\alpha\beta} = \frac{1}{2m^2_{\rm pl}}T_{\mu\nu},\end{aligned} \tag{7.6}$$

which propagates 5 polarisations of a massive spin-2 field, and whose field equations can be derived from the Fierz–Pauli action [16, 17],

$$\begin{aligned}\mathcal{S} = \tfrac{1}{2}\int {\rm d}^4x\sqrt{-g}\Big[&h^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta} + \Lambda h^{\mu\nu}\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\right)h_{\alpha\beta} \\ &- \tfrac{m^2_{\rm FP}}{2}h^{\mu\nu}\left(\delta^\alpha_\mu\delta^\beta_\nu - \overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}\right)h_{\alpha\beta}\Big].\end{aligned} \tag{7.7}$$

7.2 Perturbative Vielbein Expansion

While metric perturbations are well-known, generic vielbein perturbations are less common. The additional Lorentz fields contained in the vielbein are usually gauge-fixed, and the vielbein expanded in terms of the metric perturbation alone. However, a generic vielbein, including the Lorentz fields, can be expanded simply by,

$$e^A{}_\mu = \bar{e}^A{}_\mu + \delta e^A{}_\mu, \tag{7.8}$$

where $\bar{e}^A{}_\mu$ is a vielbein of the background metric $\bar{g}_{\mu\nu} = \bar{e}^A{}_\mu \eta_{AB} \bar{e}^B{}_\nu$, and $\delta e^A{}_\mu$ is a generic perturbation *including* the Lorentz fields. Note that $\bar{e}^A{}_\mu$ is not unique and has its own Lorentz degrees of freedom, independent of the perturbation $\delta e^A{}_\mu$.

While $g_{\mu\nu} = e^A{}_\mu \eta_{AB} e^B{}_\nu$ cannot be inverted for a general vielbein in terms of $g_{\mu\nu}$, the perturbations of $\delta e^A{}_\mu$ can be expanded in terms of the metric perturbation $h_{\mu\nu}$ and an antisymmetric tensor $\omega_{\alpha\beta}$ parametrising the Lorentz fields,

$$\delta e^A{}_\mu = \sum_{n=1}^{\infty} \delta^{(n)} e^A{}_\mu, \tag{7.9a}$$

$$\begin{aligned} \delta^{(n)} e^A{}_\mu = &\binom{1/2}{n} \bar{e}^A{}_\nu \left(h^n\right)^\nu{}_\mu \\ &+ 2 \sum_{m=1}^{n} \left(-\tfrac{1}{4}\right)^m \binom{1/2}{n-m} \bar{e}^A{}_\nu \left(\omega^m\right)^\nu{}_\lambda \left(h^{n-m}\right)^\lambda{}_\mu , \end{aligned} \tag{7.9b}$$

where,

$$\binom{1/2}{n} = \begin{cases} 1 & n = 0, \\ \frac{(-1)^{n-1}}{2^{2n-1} n} \binom{2n-2}{n-1} & n > 0, \end{cases} \tag{7.10}$$

and explicitly,

$$\delta^{(1)} e^A{}_\mu = \tfrac{1}{2} \bar{e}^A{}_\nu \Big[h^\nu{}_\mu - \omega^\nu{}_\mu \Big], \tag{7.11a}$$

$$\delta^{(2)} e^A{}_\mu = \tfrac{1}{8} \bar{e}^A{}_\nu \Big[- h^\nu{}_\sigma h^\sigma{}_\mu - 2\omega^\nu{}_\sigma h^\sigma{}_\mu + \omega^\nu{}_\sigma \omega^\sigma{}_\mu \Big]. \tag{7.11b}$$

The vielbein expansion (7.9) can be resummed to the form,

$$e^A{}_\mu = \bar{e}^A{}_\rho \left[\sqrt{\mathbb{1} + h}\,\right]^\rho{}_\sigma L^\sigma{}_\mu, \tag{7.12}$$

where $L^\mu{}_\nu = \bar{e}^\mu{}_A L^A{}_B \bar{e}^B{}_\nu$, given a Lorentz transformation $L^A{}_B$, is a congruence of $g_{\mu\nu}$, so that,

$$L^\mu{}_\alpha \bar{g}_{\mu\nu} L^\nu{}_\beta = \bar{g}_{\alpha\beta}. \tag{7.13}$$

The field $\omega_{\alpha\beta}$ is related to the Lorentz transformation by the inverse Cayley transform,

$$\omega_{\alpha\beta} = 4\bar{e}_{A\alpha}\big[(\mathbb{1}+L)^{-1}\big]^{A}{}_{B}\big[\mathbb{1}-L\big]^{B}{}_{C}\bar{e}^{C}{}_{\beta}. \tag{7.14}$$

From the form (7.12), it is easy to see that the Lorentz fields and higher order perturbations drop out of the metric,

$$g_{\mu\nu} = \bar{e}^{A}{}_{\alpha}\left[\sqrt{\mathbb{1}+h}\,\right]^{\alpha}{}_{\beta} L^{\beta}{}_{\mu}\eta_{AB}\bar{e}^{B}{}_{\rho}\left[\sqrt{\mathbb{1}+h}\,\right]^{\rho}{}_{\sigma} L^{\sigma}{}_{\nu} \tag{7.15a}$$

$$= \left[\sqrt{\mathbb{1}+h}\,\right]^{\alpha}{}_{\mu}\bar{g}_{\alpha\beta}\left[\sqrt{\mathbb{1}+h}\,\right]^{\beta}{}_{\nu} \tag{7.15b}$$

$$= \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{7.15c}$$

where the last step follows from the fact that the square root $S = \sqrt{\mathbb{1}+h}$ obeys,

$$gS = S^{\mathrm{T}}g, \tag{7.16}$$

so that any metric quantities remain independent of $\omega_{\alpha\beta}$ to all orders.

7.3 Perturbative Multi-Gravity

We now apply the formalism of vielbein perturbations to multi-gravity theories of the form (6.1) and interactions (4.24). Since the Einstein–Hilbert terms can be expressed in terms of only the metric degrees of freedom, they can be expanded by the standard methods with their individual background metrics $\bar{g}^{I}_{\mu\nu}$ and metric perturbations $h^{I}_{\mu\nu} = g^{I}_{\mu\nu} - \bar{g}^{I}_{\mu\nu}$ and individually take the form (7.2c) to quadratic order. In contrast, the multi-gravity potential is a function of the vielbeins and we must implement the perturbative formalism including the Lorentz fields. Each vielbein can be expanded in the same way, so that $e^{A}_{I\,\mu} = \bar{e}^{A}_{I\,\mu} + \delta e^{A}_{I\,\mu}$ and $\delta e^{A}_{I\,\mu}$ can be expressed in terms of the metric perturbations $h^{I}_{\mu\nu}$ and corresponding Lorentz fields $\omega^{I}_{\mu\nu}$ using (7.9b).

7.3.1 Quadratic Potential and Action

For a generic β^{IJKL}, the potential can be expanded in the form,

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \dots, \tag{7.17}$$

where,

$$V^{(0)} = \frac{2m^4}{4!}\sum_{IJKL}\beta^{IJKL}\delta^{\alpha\beta\gamma\delta}_{ABCD}\bar{e}^A_{I\,\alpha}\bar{e}^B_{J\,\beta}\bar{e}^C_{K\,\gamma}\bar{e}^D_{L\,\delta}, \tag{7.18a}$$

$$V^{(1)} = \sum_I m_I^2\sqrt{-\bar{g}_I}\,\overline{V}^I_{\mu\nu}\Big[h^{\mu\nu}_I - \omega^{\mu\nu}_I\Big], \tag{7.18b}$$

$$\begin{aligned} V^{(2)} = &-\tfrac{1}{4}\sum_I m_I^2\sqrt{-\bar{g}_I}\,\overline{V}^I_{\mu\nu}\Big[h^\mu_{I\,\sigma}h^{\sigma\nu}_I + 2\omega^\mu_{I\,\sigma}h^{\sigma\nu}_I - \omega^\mu_{I\,\sigma}\omega^{\sigma\nu}_I\Big] \\ &+\tfrac{1}{4}\sum_{IJ}\overline{W}^{IJ\,\alpha\beta}_{\mu\nu}\Big[h^\mu_{I\,\alpha}h^\nu_{J\,\beta} - 2h^\mu_{I\,\alpha}\omega^\nu_{J\,\beta} + \omega^\mu_{I\,\alpha}\omega^\nu_{J\,\beta}\Big], \end{aligned} \tag{7.18c}$$

where $\overline{V}^I_{\mu\nu}$ is given by the vielbein interaction term,

$$\begin{aligned} V^I_{\mu\nu} &= \frac{1}{2m_I^2\sqrt{-g_I}}e^A_{I\,\mu}\frac{\delta V}{\delta e^A_{I\,\alpha}}g^I_{\alpha\nu} \\ &= \frac{m^4}{3!\,m_I^2\sqrt{-g_I}}\sum_{JKL}\beta^{IJKL}\delta^{\alpha\beta\gamma\delta}_{ABCD}e^A_{I\,\mu}e^B_{J\,\beta}e^C_{K\,\gamma}e^D_{L\,\delta}g^I_{\alpha\nu}, \end{aligned} \tag{7.19}$$

evaluated on the background vielbeins $\bar{e}^A_{I\,\mu}$, and we have defined the quantity,

$$\overline{W}^{IJ\,\alpha\beta}_{\mu\nu} = \frac{2m^4}{3!}\sum_{KL}\beta^{IJKL}\delta^{\alpha\beta\gamma\delta}_{ABCD}\bar{e}^A_{I\,\mu}\bar{e}^B_{J\,\nu}\bar{e}^C_{K\,\gamma}\bar{e}^D_{L\,\delta}, \tag{7.20}$$

which, if we restrict to Hassan–Schmidt–May interaction ($\beta^{IJKL} = \beta^I\beta^J\beta^K\beta^L$), simplifies to,

$$\overline{W}^{IJ\,\alpha\beta}_{\mu\nu} = \frac{m^4}{3}\beta^I\beta^J\delta^{\alpha\beta\gamma\delta}_{ABCD}\bar{e}^A_{I\,\mu}\bar{e}^B_{J\,\nu}\bar{u}^C_{\ \gamma}\bar{u}^D_{\ \delta}, \tag{7.21}$$

and $\bar{u}^A_{\ \alpha}$ is the invertible background quantity,

$$\bar{u}^A_{\ \mu} = \sum_{I=1}^{\mathcal{N}}\beta^I\bar{e}^A_{I\,\mu}. \tag{7.22}$$

By combining (7.2c) and (7.18c), one can now write down the quadratic Lagrangian,

$$\begin{aligned} \mathcal{L}^{(2)} = \sum_I m_I^2\sqrt{-\bar{g}_I}\Big[&\tfrac{1}{2}h^{\mu\nu}_I\Big({}^I\mathcal{E}_{\mu\nu}{}^{\alpha\beta} - \tfrac{1}{2}\big(\bar{g}^I_{\mu\nu}\overline{R}^{\alpha\beta}_I - (\overline{R}_I - 2\Lambda_I)\delta^\alpha_\mu\delta^\beta_\nu\big)\Big)h^I_{\alpha\beta} \\ &- \overline{V}^I_{\mu\nu}\Big(3h^\mu_{I\,\sigma}h^{\sigma\nu}_I - h_I h^{\mu\nu}_I - 2\omega^\mu_{I\,\sigma}h^{\sigma\nu}_I + \omega^\mu_{I\,\sigma}\omega^{\sigma\nu}_I\Big) \\ &- \tfrac{1}{4m_I^2\sqrt{-\bar{g}_I}}\sum_J\overline{W}^{IJ\,\alpha\beta}_{\mu\nu}\Big(h^\mu_{I\,\alpha}h^\nu_{J\,\beta} - 2h^\mu_{I\,\alpha}\omega^\nu_{J\,\beta} + \omega^\mu_{I\,\alpha}\omega^\nu_{J\,\beta}\Big) \\ &+ \Big(\overline{G}^I_{\alpha\beta} + \Lambda_I\bar{g}^I_{\alpha\beta} + \overline{V}^I_{\alpha\beta}\Big)\Big(h^\alpha_{I\,\sigma}h^{\sigma\beta}_I - \tfrac{1}{4}h_I h^{\alpha\beta}_I\Big)\Big], \end{aligned} \tag{7.23}$$

where the last line vanishes if we impose the background field equations.

7.3.2 Proportional Backgrounds

We now restrict to Hassan–Schmidt–May theory, where the parameters β^{IJKL} decompose as $\beta^I\beta^J\beta^K\beta^L$ and the potential takes the form (6.2), and focus on a class of background solutions for which the mass eigenfields and the corresponding Fierz–Pauli mass terms can be identified.

Consider an Ansatz where all vielbeins are proportional,

$$e_{I\ \mu}^{A}(x) = c_I \bar{e}^{A}_{\ \mu}(x), \tag{7.24}$$

for some non-vanishing real scalar constants c_I, and $\bar{e}^{A}_{\ \mu}$ is a common background vielbein. The vacuum background field equations (6.7) reduce to,

$$\overline{G}_{\mu\nu} + \Big[\Lambda_I c_I^2 + \beta^I \tfrac{m^4 k^3}{m_I^2 c_I}\Big]\bar{g}_{\mu\nu} = 0, \tag{7.25}$$

where $k = \sum_I \beta^I c_I$ and $\overline{G}_{\mu\nu}$ is the Einstein tensor of the background metric $\bar{g}_{\mu\nu}$ constructed from $\bar{e}^{A}_{\ \mu}$.

Since the Einstein tensor is invariant under a constant scaling of the metric, $\overline{G}_{\mu\nu}$ is the same for each sector, and combining the field equations (7.25) yields the consistency conditions,

$$\Lambda_1 c_1^2 + m^4 k^3 \frac{\beta^1}{m_1^2 c_1} = \ldots = \Lambda_{\mathcal{N}} c_{\mathcal{N}}^2 + m^4 k^3 \frac{\beta^{\mathcal{N}}}{m_{\mathcal{N}}^2 c_{\mathcal{N}}}, \tag{7.26}$$

on the proportionality constants c_I. This is a function of the combinations $\Lambda_I c_I^2$, $\beta^I c_I$, and $m_I^2 c_I^2$, and is invariant under the rescaling,

$$\beta^I \mapsto \lambda \beta^I, \tag{7.27a}$$
$$m_I \mapsto \lambda m_I, \tag{7.27b}$$
$$\Lambda_I \mapsto \lambda^2 \Lambda_I, \tag{7.27c}$$
$$c_I \mapsto \lambda^{-1} c_I, \tag{7.27d}$$

with some arbitrary λ, implying that we cannot determine all the c_I from (7.26). We will use this freedom to initially fix $c_1^{-1} = \beta^1$ without loss of generality.

If we introduce the notation $y_I = \beta^I c_I$ (note that $y_1 = 1$), the consistency conditions (7.26) can be expressed in the form of polynomials in y_I,

$$\mathcal{P}_I = m^4 \Big[\sum_{J=1}^{\mathcal{N}} y_J\Big]^3 \Big[\tfrac{\beta_1^2}{m_1^2} y_I - \tfrac{\beta_I^2}{m_I^2}\Big] + y_I \Big[\tfrac{\Lambda_1}{\beta_1^2} - \tfrac{\Lambda_I}{\beta_I^2} y_I^2\Big], \tag{7.28}$$

for $I > 1$ and we have written $\beta^I = \beta_I$ for notational convenience. These are $\mathcal{N}-1$ coupled fourth-order polynomials in $\mathcal{N}-1$ variables. Solutions to such systems of equations can be complicated and the solution space generically ranges from the null set to uncountably many solutions. However, since the I^{th} polynomial $\mathcal{P}_I$ is strictly negative if $y_I = 0$ and $y_{J\neq I} > 0$,

$$\mathcal{P}_I(y_I = 0, y_{J\neq I} > 0) = -m^4 \frac{\beta_I^2}{m_I^2}\Big[\sum_{J=1}^{\mathcal{N}} y_J\Big]^3 < 0, \tag{7.29}$$

and for sufficiently large positive y_I, the dominant 4^{th} order term is positive,

$$\mathcal{P}_I(y_I \gg y_J) = m^4 \frac{\beta_1^2}{m_1^2}\Big[\sum_{J=1}^{\mathcal{N}} y_J\Big]^3 y_I + \ldots > 0, \tag{7.30}$$

$\mathcal{P}_I$ must change sign in the positive quadrant $y_J > 0$. Since this is true for all $\mathcal{P}_I$ it follows from the Poincaré–Miranda theorem that all the $\mathcal{P}_I = 0$ surfaces in that quadrant necessarily intersect in at least one point.[1] This shows that there always exist solutions for c_I so that $y_I = \beta^I c_I$ are positive.

However, finding explicit solutions $c_I(\beta^J, m_J, \Lambda_J)$ is complicated. If we reintroduce c_1 as a free parameter and instead choose to eliminate the theory parameters β^I, parametric solutions can be found explicitly,

$$\beta^I = \frac{m_I^2 c_I}{m} \frac{\Lambda - \Lambda_I c_I^2}{\big[\sum_{J=1}^{\mathcal{N}} m_J^2 c_J^2(\Lambda - \Lambda_J c_J^2)\big]^{3/4}}, \tag{7.31}$$

where Λ is an arbitrary parameter which also corresponds to the cosmological constant of the background equation (7.25), so that,

$$\overline{G}_{\mu\nu} + \Lambda \overline{g}_{\mu\nu} = 0. \tag{7.32}$$

Thus, the Hassan–Schmidt–May field equations admit solutions corresponding to Einstein spacetimes parametrised by a cosmological constant Λ, if the β-parameters fulfil (7.31).

7.3.3 Mass Spectrum

Around the proportional background solutions above, one can identify the mass spectrum of multi-gravity, by first noting that the linear Lorentz constraints reduce to,

$$\epsilon^{\alpha\beta\gamma\delta}\Big[\sum_J \beta_J c_J\big(\omega^I_{\alpha\beta} - \omega^J_{\alpha\beta}\big)\Big]\overline{u}^{[A}_{\ \gamma}\overline{u}^{B]}_{\ \delta} = 0, \tag{7.33}$$

[1] Poincaré–Miranda theorem: If $P_I : [a_1, b_1] \times \ldots \times [a_{\mathcal{N}}, b_{\mathcal{N}}] \to \mathbb{R}$ are continuous, and if for each $I = 1, \ldots, \mathcal{N}$ one has $P_I(y_1, \ldots, a_I, \ldots, y_{\mathcal{N}}) \leq 0$ and $P_I(y_1, \ldots, b_I, \ldots, y_{\mathcal{N}}) \geq 0$ for all $y_{J\neq I}$ in the domain, then there exists $x \in [a_1, b_1] \times \cdots \times [a_{\mathcal{N}}, b_{\mathcal{N}}]$ such that $P_I(x) = 0$ for all I.

which, since the background vielbein sum $\overline{u}^A_{\ \mu}$ (7.22) is invertible, have the trivial solutions $\omega^I_{\alpha\beta} = \omega^J_{\alpha\beta}$. With these solutions the Lorentz fields drop out of the quadratic action, and (7.23) simplifies to,

$$\mathcal{S}^{(2)} = \int \mathrm{d}^4x\sqrt{-\overline{g}} \sum_{I=1}^{\mathcal{N}} \Big[\tfrac{1}{2} h_I^{\mu\nu} \overline{\mathcal{E}}_{\mu\nu}{}^{\alpha\beta} h^I_{\alpha\beta} + \tfrac{\Lambda}{2} h_I^{\mu\nu} \left(\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2} \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \right) h^I_{\alpha\beta} \\ - \tfrac{1}{4} \left(\delta^\alpha_\mu \delta^\beta_\nu - \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \right) \sum_{J=1}^{\mathcal{N}} h_I^{\mu\nu} \mathbb{M}_{IJ} h^J_{\alpha\beta} \Big], \tag{7.34}$$

where $\overline{\mathcal{E}}_{\mu\nu}{}^{\alpha\beta}$ is the differential operator (7.3) constructed from the background metric $\overline{g}_{\mu\nu}$ and its Levi-Civita connection $\overline{\nabla}_\mu$.

The second row of (7.34) has the correct tensor structure to be a Fierz–Pauli mass term (7.6), but is currently in non-diagonal form since the mass matrix,

$$\mathbb{M}_{IJ} = m^4 k^2 \left[\delta_{IJ} \left(\frac{\beta^I}{m_I^2 c_I} (k - \beta^I c_I) + \frac{\beta^I \beta^J}{m_I m_J} \right) - \frac{\beta^I \beta^J}{m_I m_J} \right]. \tag{7.35}$$

is non-diagonal.

The matrix $\mathbb{M}$ can be written as rank-1 deformation $v_I v_J$ of a diagonal matrix $\mathbb{D}$, so that,

$$\mathbb{M} = \mathbb{D} - \mathbf{v}\mathbf{v}^{\mathrm{T}}, \tag{7.36a}$$

$$\mathbb{D} = \mathrm{diag}(d_1, \dots, d_{\mathcal{N}}), \qquad d_I = m^4 k^3 \frac{\beta^I}{m_I^2 c_I}, \tag{7.36b}$$

$$\mathbf{v} = m^2 k (\beta^1/m_1, \dots, \beta^{\mathcal{N}}/m_{\mathcal{N}})^{\mathrm{T}}. \tag{7.36c}$$

The eigenvalues and vectors of such matrices can generically not be determined explicitly, but given implicit eigenvalues μ_I^2 determined by the roots of the characteristic equation,

$$\mathfrak{p}(\mu^2) = -\frac{\mu^2}{k} \sum_{I=1}^{\mathcal{N}} \beta_I c_I \prod_{\substack{J \neq I \\ J=1}}^{\mathcal{N}} \left(d_J - \mu^2 \right) = 0, \tag{7.37}$$

the eigenvectors take the form,

$$f^I_{\mu\nu} = \frac{1}{|\mathbf{x}_I|} \sum_{J=1}^{\mathcal{N}} \frac{m_J c_J}{1 - \frac{\mu_I^2}{d_J}} h^J_{\mu\nu}, \qquad |\mathbf{x}_J| = \sqrt{\sum_I \frac{m_I^2 c_I^2}{\left(1 - \frac{\mu_J^2}{d_I}\right)^2}}. \tag{7.38}$$

These solutions diagonalise the quadratic action (7.34),

$$\mathcal{S}^{(2)} = \int \mathrm{d}^4x\sqrt{-\overline{g}} \sum_{I=1}^{\mathcal{N}} \Big[\tfrac{1}{2} f_I^{\mu\nu} \mathcal{E}_{\mu\nu}{}^{\alpha\beta} f^I_{\alpha\beta} + \tfrac{\Lambda}{2} f_I^{\mu\nu} \left(\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2} \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \right) f^I_{\alpha\beta} \\ - \tfrac{\mu_I^2}{4} f_I^{\mu\nu} \left(\delta^\alpha_\mu \delta^\beta_\nu - \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \right) f^I_{\alpha\beta} \Big], \tag{7.39}$$

which now takes the form of $\mathcal{N}$ copies of the Fierz–Pauli action (7.7) corresponding to a theory of $\mathcal{N}$ spin-2 fields $f^I_{\mu\nu}$ with masses μ_I^2, confirming that Hassan–Schmidt–May theory is a theory of $\mathcal{N}$ non-linearly interacting spin-2 fields. The field equations for each perturbation $f^I_{\mu\nu}$ now take precisely the form (7.6), with the masses given by solutions to (7.37).

If all $\beta^I \neq 0$, then the eigenmass equation (7.37) has exactly one zero root, $\mu^2 = 0$. Consequently, the spectrum contains one massless spin-2 field, while the remaining $\mathcal{N} - 1$ modes are massive. If instead $\beta_K = 0$ for some K, then the K-th vielbein does not enter the interaction potential and therefore decouples from the other sectors. The corresponding perturbation $h^K_{\mu\nu}$ is then an additional free, non-interacting massless spin-2 field.

The massless perturbation in the interacting sector can be written explicitly,

$$f^{(\mu^2=0)}_{\mu\nu} = \frac{1}{\sqrt{\sum_I m_I^2 c_I^2}} \sum_{J=1}^{\mathcal{N}} m_J c_J h^J_{\mu\nu}\,, \tag{7.40}$$

and is a linear combination of all perturbations $h^J_{\mu\nu}$ since $m_J c_J \neq 0$. Consequently, each vielbein (and its associated metric) entering the non-linear action (6.1) receives contributions from the massless and the massive modes after diagonalisation. In particular, if the Standard Model couples to a vielbein $e_I{}^A{}_\mu$, then the corresponding "physical" metric, whose geometry we experience, contains both the massless and massive spin-2 excitations.

For $\mathcal{N} > 4$, the eigenmasses, i.e. the roots of (7.37) can only be obtained numerically, but given that the background consistency conditions are solved so that $\beta^I c_I > 0$, their structure can be shown to fulfil the bounds,

$$0 = \mu_1^2 < d_1 \leq \mu_2^2 \leq d_2 \leq \ldots \leq d_{\mathcal{N}-1} \leq \mu_{\mathcal{N}}^2 \leq d_{\mathcal{N}}\,, \tag{7.41}$$

where we have relabelled the fields $f^I_{\mu\nu}$ so that the diagonal elements $d_I = \beta_I m^4 k^3 / m_I^2 c_I$ of $\mathbb{D}$ are in increasing order $d_I \leq d_{I+1}$. This shows that the mass spectrum is bounded from above by $d_{\mathcal{N}}$ and from below by 0, implying a non-tachyonic spectrum.

Equation (7.41) is a complicated expression in terms of β_I, m_I, Λ_I due to the $d_I(c_I)$ dependence of the solutions $c_I(\beta_I, m_I, \Lambda_I)$ of (7.28). If we instead use the parametric solutions (7.31), the condition can be expressed in terms of only free theory parameters Λ, Λ_I and c_I,

$$0 = \mu_1^2 < \Lambda - \Lambda_1 c_1^2 \leq \ldots \leq \mu_{\mathcal{N}}^2 \leq \Lambda - \Lambda_{\mathcal{N}} c_{\mathcal{N}}^2, \tag{7.42}$$

so that the mass spectrum in principle can be tuned, at least within the range $[\Lambda - \Lambda_I c_I^2, \Lambda - \Lambda_{I+1} c_{I+1}^2]$.

In the above form, it is also evident that if all the "bare" cosmological constants Λ_I are zero, the mass spectrum becomes degenerate and $\mu_I^2 = \Lambda$, and that for Minkowski backgrounds ($\Lambda = 0$), $\Lambda_I < 0$. On de Sitter backgrounds, one can show that one can always avoid the Higuchi ghost [82], as long as,

$$\Lambda_1 c_1^2 < \tfrac{1}{3}\Lambda. \tag{7.43}$$

On anti-de Sitter, negative masses are in principle allowed, as long as the Breitenlohner–Freedman bound [83],

$$\mu_I^2 > \tfrac{3}{4}\Lambda, \tag{7.44}$$

is fulfilled. Since the smallest eigenvalue of $\mathbb{M}$ is always bounded by $d_1 - |\mathbf{v}|^2$ [84], a sufficient condition,

$$\frac{\Lambda}{4} > \Lambda_1 c_1^2 + \frac{1}{m^4 k^4} \sum_{I=1}^{\mathcal{N}} m_I^2 c_I^2 \left(\Lambda - \Lambda_I c_I^2\right)^2, \tag{7.45}$$

avoids any pathological masses on anti-de Sitter.

The perturbative analysis thus confirms, at the linearised level, the results of the non-linear constraint analysis of Chapter 6: the Hassan–Schmidt–May theory propagates one massless and $\mathcal{N}-1$ massive spin-2 fields with no additional scalar ghost-modes, but also that the mass spectrum is non-tachyonic and that other known instabilities can always be avoided.

Chapter 8

Summary and Outlook

8.1 Summary

This thesis has covered central aspects of multi-gravity theories: analysing their constrained dynamics, determining when they are ghost-free, and characterising their mass spectrum, using the formal developments and results of Papers I–IV.

Using the unimodular decomposition, we isolated the origin of the Boulware–Deser ghost in theories where multiple spin-2 fields with Einstein–Hilbert kinetic terms are coupled by non-derivative interactions. The would-be Boulware–Deser ghosts are tied to the constraints associated with the lapse variables, and consistency requires enough constraints to remove all such pathological modes. Employing the ghost-isolating Ansatz, the canonical analysis becomes transparent and from this, we derived the necessary condition that the interaction potential must be linear in each lapse in order for the full set of ghost-eliminating constraints to exist. Imposing this condition together with local Lorentz covariance and the requirement that the interaction is a scalar density singles out the Hinterbichler–Rosen form (4.24) [64] as the unique class of admissible vielbein potentials.

In Chapter 5 we showed that this class of potentials is not ghost-free for arbitrary interaction parameters β^{IJKL}, since the Lorentz constraints generically reintroduce lapse dependence and destroy the ghost-eliminating constraints. For irreducible interactions with $\mathcal{N} \geq 3$, we derived the necessary condition that the parameters must factorise as $\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L$, corresponding to the Hassan–Schmidt–May determinant coupling, while for $\mathcal{N} = 2$ the parameters are unconstrained and reproduce Bimetric Theory. Reducible interactions were argued to be trees of these irreducible building blocks, leading to the conjecture that tree-structured combinations of Bimetric and determinant interactions constitute the most general ghost-free non-derivative multi-gravity theory.

In Chapter 6, using the canonical framework developed in Paper II, we reviewed the Hassan–Schmidt–May theory and its Hamiltonian formulation, summarising its canonical structure, constraints, and the construction of the symmetry generators. The key result of Paper III was the explicit derivation of the ghost-eliminating constraints in the equal-boost Ansatz, where the constraint algebra and Poisson brackets become tractable. With the constraints identified and classified, the physical phase space was shown to contain exactly the degrees of freedom of one massless and $\mathcal{N}-1$ massive spin-2 fields, i.e. $2+5(\mathcal{N}-1)$ propagating helicities, with no additional scalar ghost-modes.

In Chapter 7, we presented the perturbative expansion of a vielbein in terms of independent metric and Lorentz-field perturbations from Paper IV, and used it to expand the multi-gravity interaction potential to quadratic order. This perturbative vielbein formalism was applied to expand the Hassan–Schmidt–May theory around proportional Einstein backgrounds. Given the background consistency conditions, the quadratic action can be diagonalised into one massless and $\mathcal{N}-1$ massive spin-2 sectors. The corresponding mass eigenstates were obtained implicitly through the structure of the quadratic mixing, and the resulting mass spectrum was shown to be non-tachyonic and to avoid known instabilities that arise from massive spin-2 fluctuations on maximally symmetric Einstein spacetimes, thereby establishing the viability of the proportional-background branch at the linearised level.

Together with the necessary conditions derived in Paper I, the results of this thesis yield a strong restriction on the space of consistent non-derivative multi-gravity theories of the form (4.1), and identify the Hassan–Schmidt–May theory and its reducible extensions as the unique candidates for ghost-free multi-gravity. Several important questions, however, remain open.

8.2 Outlook

Perhaps the most pressing open problem is the completion of the proof that the Hassan–Schmidt–May theory is ghost-free beyond the equal-boost Ansatz. While Paper III established the existence of the ghost-eliminating constraints under this simplifying assumption, a proof valid for fully generic field configurations remains to be formulated. The canonical formalism developed in Papers II and III, in particular the Lorentz covariant vielbein decomposition, provides the necessary tools, but the explicit computation of the Poisson brackets for arbitrary boost and rotational fields is technically demanding.

A related open problem is the formulation of the Hassan–Schmidt–May theory purely in terms of metrics. For Bimetric Theory, the Lorentz constraints can be solved explicitly via the symmetrisation condition (5.15), allowing the vielbeins to be eliminated in favour of the square root matrix (5.16). For the multivielbein theory, the generalised Lorentz constraints (6.10) do not have a known closed-form solution, and a metric formulation analogous to the Bimetric one has not yet been constructed. Such a formulation would be valuable both for practical computations and for a deeper understanding of the geometric content of the theory.

On the phenomenological side, the exploration of solutions beyond the proportional backgrounds considered in Chapter 7 is still in its early stages. Cosmological solutions, the resulting modified Friedmann equations, and the associated Bianchi constraints ${}^{I}\nabla_{\mu} V_{I}^{\mu\nu} = 0$ remain to be analysed systematically. The presence of multiple massive spin-2 fields with a tuneable mass spectrum opens the possibility for phenomenological applications, including massive spin-2 fields as dark matter candidates or multi-gravity modifications as a source of dark energy, but concrete predictions require further work.

A more fundamental question concerns the multiplet structure of spin-2 fields. One of the original motivations for studying multi-gravity theories is the analogy with the Standard Model, where fields of identical spin are arranged in gauge multiplets that transform into each other under symmetry transformations. The theories studied in this thesis do not yet exhibit such structures: the diagonal Lorentz and diffeomorphism symmetries do not mix the different vielbein sectors. For a genuine multiplet structure to emerge, one would need symmetry transformations that rotate the vielbeins into each other, analogous to how the electroweak SU(2) rotates the gauge bosons. However, any such mixing of the vielbeins would necessarily also mix the corresponding Einstein–Hilbert kinetic terms, introducing derivative interactions between the spin-2 fields. The theories considered here, with purely non-derivative interactions and separate Einstein–Hilbert terms, are therefore structurally limited in this respect. Exploring derivative interactions and modified kinetic terms is a natural next step, though the constraint analysis, already demanding for non-derivative interactions, becomes substantially more involved when the kinetic structure itself is modified.

As has been evident throughout this thesis, and in the historical development of both General Relativity and the Standard Model, theoretical consistency is a remarkably powerful tool for constraining and in some cases essentially determining the form of physical interactions. The results presented here suggest that this pattern extends to the spin-2 sector, and that further exploration of multi-gravity theories, guided by the same principle, is a promising direction for future work.

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Part II