

Taking on board a CME perspective in early algebra

The examination in Paper 3 made evident that research on early algebra so far has privileged the logical and psychological interpretations, using Skovsmose and Valero's (2008) categories, of how knowing algebra can be powerful mathematical ideas for students. It also showed that the cultural and sociological interpretations were scarce. The question emerges: What would it mean to take on board CME to expand possible ways in which early algebra can be thought as powerful mathematical ideas for students culturally and sociologically speaking?

It is important to note that the conceptualization that follows here did not precede the empirical experimentation that unfolds in the third part of my research. Rather, it emerged as I engaged in the process of writing this manuscript and of further making sense of how CME could inform the work in my licenciante thesis (Papers 1 and 2) and support the direction initiated in Paper 3. Therefore, the sections below illuminate the issue of conceptually expanding views of mathematics and mathematics education, implicitly adopted through my involvement in the field of early algebra from a LAT perspective, when reconsidered through the lens of CME. Therefore, in what follows, I will begin with a brief discussion of *CME* as an orienting framework. I continue by addressing the notion of critical in relation to my research and the new understandings that critical in CME brings to think of mathematics education and school algebra. I explore this by focusing on mathematics as non-neutral and in action, mathematics and mathematics education in context, and mathematics education for citizenship.

These are orienting ideas that, when contextualized to early algebra in the socio-ecological, help me address my second research question of how early algebraic ideas can be understood for the combined purpose of developing younger students' algebraic thinking and fostering citizenship. I conclude the chapter by discussing how these considerations underscore the necessity of the work presented in chapters 4 and 5.

Critical Mathematics Education as an orienting framework

CME has been developed since the early 1990s as a recognized area of research and practice in mathematics education (e.g., Skovsmose & Nielsen, 1996; Andersson & Barwell, 2021). It has developed in several parts of the world in relation to the so called social and socio-political turn in mathematics education (e.g., Gutierrez, 2013). Drawing on Skovsmose (2023a), CME can be characterized as an approach to mathematics education both as research and practices of teaching and learning, with certain concerns and hopes. CME is concerned with what happens in classrooms, not only in terms of what mathematics is being learned but also on the type of interactions that can help students becoming citizens. It is also concerned with the lives of students outside the mathematics classrooms and how economic inequalities and exploitation, differentiation, and racism impact them. It is also concerned with the environment and how human action has had a devastating impact on it. Finally, it is also concerned with mathematics itself “not only as a school subject but also as applied [in] a variety of contexts” (Skovsmose, 2023a, p. 6). CME also embraces the hope that many of the situations of concern mentioned before can be different. It means that there is an engagement in research and also practice to transform and imagine new forms of mathematics and mathematics education to contribute to social change (p. 9).

Well aware of the variety of directions of CME, in my work I have found productive to take some of Skovsmose’s (1994, 2023a) considerations to further expand some of the assumptions and foundations of early algebra and my previous work, such as the notion of critical, mathematical as non-neutral and in action, mathematics and mathematics education in context, and mathematics education for citizenship in students life-worlds.

Expanding the notion of *critical*

In the licentiate thesis, the notion of critical, following LAT and Variation theory, was used in three senses. First, it emphasized the importance of identifying the mathematical elements that are pivotal for algebraically describing and justifying generalizations of patterns, which are at the core of early algebra (Paper 1). Second, critical referred to the importance of teachers' pedagogical intention in supporting students' reflection, argumentation, and reasoning about mathematical structures (Paper 2). Third, it referred to supporting students' understanding of key concepts of early algebra and their connection to other mathematical objects within algebra itself (papers 1 and 2).

When incorporating CME, a broader notion of critical becomes adopted that goes beyond the important, key aspects of mathematical thinking and connects mathematics with society. This connection must be examined and addressed (Skovsmose, 2023a). I therefore explore the expansion of the notion of critical in terms of mathematics as non-neutral and in action, mathematics education in context, and mathematics education for citizenship. Although these three aspects differ, they are closely interconnected. Therefore, I distinguish them analytically, well aware that my explanations may overlap.

Mathematics as non-neutral and in action

In the licentiate thesis, mathematics was viewed as both objective and neutral—yet socially and culturally constructed, according to the theoretical assumptions of LAT. Where mathematics concerned early algebra, the focus was on how number patterns comprise internal mathematical relationships. These interrelationships were constituted by mathematical structures that represented quantitative relationships between the position of a number and its value in a number pattern. These relationships were also general, meaning they were consistent across all numbers in the given number pattern. Additionally, these internal relationships were situated within the context of the number patterns themselves. That means

that they were conceived as being situated in a pure mathematical context. Consequently, in the absence of contexts other than purely mathematical, these interrelationships may be perceived by students as objective and neutral.

This conception of mathematics aligns with LAT and Davydov's understanding of the essence of mathematics and of a mathematical concept lying in its internal logic and in its relationships with other mathematical concepts. Both mathematics and the concepts that constitute it, although formed in cultural-historical processes, appear as objects with an existence of their own, independent of the people who have formulated them, who will appropriate them or who will later use them. They need to be activated in student's mathematical thinking and reflection.

Skovsmose (2023b) presents an argument that highlights the performative aspects of mathematics from a broader perspective, pointing to why mathematics might not be viewed as objective and neutral. He does this by addressing how mathematics not only forms our view of the world but also forms the world itself:

A basic idea associated with a performative interpretation of mathematics is that mathematics is not a transparent descriptive tool by means of which we may picture reality. Mathematics forms what we pay attention to, and what we might ignore. Mathematics forms our world views. It forms our world as well. (Skovsmose, 2023b, p. 140)

What does it mean to say that mathematics forms our world views? To address this question, we need to examine what it means to say that mathematical concepts and models are abstractions. Skovsmose (1994) distinguishes three types of abstractions connected to mathematics: *thinking abstraction*, *realized abstraction*, and *real abstraction*. To clarify these three concepts, I present an example drawn from Hall and Barwell (2021), who employ Skovsmose's notions to discuss the origins and effects of the Body Mass Index (BMI) as a model of the phenomenon of obesity. This excerpt illustrates how obesity constitutes a realized abstraction:

Our point, then, is not that obesity does not exist or that obesity is not associated with health risks. Our point is that the certainty of science, through the use of mathematics, turns a fuzzy and complex phenomenon into a normative, prescriptive, and ideological abstraction, which in turn leads to concrete interventions (i.e., control), in the form of advice, medication, and penalties, such as, in some cases, exclusion from school. This normativity, in turn, is likely to feed into wider discourses relating to such topics as body image, femininity, masculinity, identity, and race. (p. 222)

These authors highlight how a mathematical model is an abstraction of a defined reality that is useful for approaching it. Models select features of reality; in other words, there might be other features that could be selected, and then a model would reveal an alternative reality.

Returning to Skovsmose's (1994) three notions of abstraction, a *thinking abstraction* involves using mathematical concepts, such as mathematical models, to provide an image of "reality" or of "phenomena" such as obesity, climate change or a pandemic. However, this image only shows a selection of features that the reality consists of. *Realized abstraction* refers to what is perceived as reality once thinking abstractions begin to be applied and we start operating with them. In other words, the reality that is perceived is influenced by which features are selected to be part of a mathematical model. *Real abstractions* concern how, once we start perceiving in a particular way thanks to the use of realized abstractions, the things that the mathematical models allowed one to perceive, in a sense, begin to become real. This means that what is perceived as reality is influenced by how thinking and realized abstractions mediate people's perception of the world. In this way, what mathematical concepts and models have prioritized shapes, for example, what we perceive as crucial aspects of obesity or climate change today and how they inform our daily actions. Because these different abstractions form our sense of reality, then it can be said that mathematical concepts and models are put into *action*. Thus, Skovsmose argues, mathematical abstractions are not merely concepts existing as neutral, objective, universal, and value-free entities.

Skovsmose (1994) furthermore formulates the notion of *the formatting power of mathematics* to call attention to the need for a critical awareness of the “invisible” power that mathematics can exercise as it acts. This idea is most effectively understood when mathematics is examined within the context of broader social structures rather than considered as an independent body of knowledge. As mathematics often appears integrated into practical applications such as algorithms, tables, and graphs, it has an implicit and occasionally obscured influence. As a result, its impact often remains unnoticed, yet it can shape our perceptions and reactions to significant social, political, scientific, technological, or socio-ecological issues, which are frequently addressed through mathematical models.

Recently, Skovsmose (2023b) reformulated this idea as mathematics being agentic and performative. The argument highlights the significance of mathematics in our world. It is important to note that this does not imply that mathematics is “bad.” Instead, it suggests that it exercises power, which means that it cannot be viewed solely as neutral or detached from the situations where it operates or the people who use it or the interests that it may help privilege. Skovsmose (2024) addresses this by pointing out that: “[...] mathematics is [...] also indefinite concerning power. Mathematics can associate itself with any form of power structure; it is not particularly selective. Mathematics appears to be ready to serve anyone and anything” (Skovsmose, 2024, p. 125).

For instance, the use of mathematical models that incorporate statistical data can influence policy decisions as well as individual behaviors, demonstrating how mathematical descriptions have real-world ethical consequences. This means that mathematics in action can critically influence different kinds of choices: “dangerous, courageous, risky, harmless, benevolent, praiseworthy, etc.” (Skovsmose, 2011, p. 70). Here, the term “critically” refers not to criticism of mathematics itself, but to the thoughtful examinations of the values and assumptions embedded in mathematical models and the choices they influence. In essence, the formatting power of mathematics and its performativity are realized through how the

three forms of mathematical abstractions explored above are used in society.

The idea of mathematics as non-neutral and in action makes visible the importance of “context.” I therefore explore what mathematics education in context can mean.

Mathematics and mathematics education in context

As previously stated, the conception of mathematics in LAT places mathematical concepts in the context of mathematical structures. Radford (2021, p. 327) argues that the view of mathematics as neutral, objective is connected to a scientific outlook on the world where “theoretical scientific thought is considered the pinnacle of human cognition.” Such a scientific outlook was central to LAT and Davydov’s understanding of the essence of a mathematical concept. Radford (2021) criticizes such a view and advances a historical argument to demonstrate why Davydov’s outlook on the world may be insufficient today:

We cannot fail to see that Davydov’s concept of the concept is imbued with a general outlook of the world that favors a particular form of knowability—the one that predominated throughout the twentieth century and considered scientific theoretical thought as the summit of human cognition. Scientific theoretical thought was perceived as the model par excellence for understanding the world. (Radford, 2021, p. 338)

Radford highlights the fact that Davydov’s definition of a concept is tied to his cultural context in Russia during the mid-twentieth century. The definition of a concept has also been influenced by societal and cultural trends regarding the type of knowledge required at the time to understand the world (e.g., Freiman & Fellus, 2021). There was a need to associate mathematics with scientific theoretical thought in Davydov’s societal, cultural, and historical context.

Skovsmose (2024) also draws attention to history, placing particular emphasis on how mathematics may serve different interests in different times:

Mathematics does not reflect any timeless realities, but is part of complex human constructions that never reach a final form. Mathematics gets formed through ongoing processes of construction, demolition, renovation, and reconstruction. All mathematical entities and theories can be different; mathematics can always find new and, at times, excellent applications; and mathematics may come to serve any interest. (Skovsmose 2024, viii)

This citation prompts consideration of how mathematical concepts do not have a fixed essence—criticizing Davydov’s view from today’s perspective. Rather, they are shaped by societal shifts in knowledge deemed essential for understanding the world at a given time. In other words, in Davydov’s time, there was a need to have a general scientific outlook on the world. However, today the world is characterized by uncertainty (e.g., Skovsmose, 2014), suggesting that the narrow scientific outlook on the world, embedded in mathematical concepts, may be insufficient. Indeed, Radford concurs as he argues for the need to broaden our view of concepts:

Yet, I prefer not to conceive of a concept as a mental rule, but as something more *poetic*, something that brings together the cultural rationality and worldview of the contexts where the object has emerged and evolved with all its historical and political tensions. In this view a concept would be what we *enact* with *others* in joint activity—a cultural-historical enactive experience that is not merely conceptual and theoretical, but also esthetic, ethical, political, and emotional; something that *questions* us. (Radford, 2021, p. 341)

In this citation, Radford refers to entities not being in the minds of students and suggests seeing them as situated in people’s activity together in a broader field of experience. This invites us to bring together a worldview that is dependent on the context in which these entities have emerged and evolved historically, and in the

current society, also within socio-ecological tensions. More precisely, one could consider that mathematical concepts have emerged in people's activity together in broader contexts. This leads us to connect to Skovsmose's formulations on mathematical concepts as performative and formatting abstractions.

What does the inseparability of mathematical concepts and context mean for mathematics education? I begin by addressing Skovsmose's notion of *critical situations* (2023a), which offers the potential to bring together the notions of critical, abstraction, and context. Skovsmose (2023a) refers to an example that Gutstein (2018) worked with his students to examine the high costs of housing in relation to average family income in their community, as many families were unable to pay and mortgages and their houses were foreclosed.

Foreclosures are critical situations where mathematics becomes real. We are not only dealing with mathematical calculations that could be done correctly or not; we are dealing with the formation of real-life conditions experienced by many families. We are dealing with realisations. (Skovsmose, 2023a, p. 99)

In the quote above, Skovsmose refers to a foreclosure as a situation that is critical because it is real and affects a family's life conditions in very concrete and serious ways. Calculations on what income is needed to pay a mortgage with a certain percentage of interest rate becomes in this case a critical situation for students. So is the case with many of the prevailing socio-ecological crises confronting our global society, and how they affect all of us and our life conditions. The notion of critical situations elucidates the connection between a socio-ecological issue and the way mathematical models depict it, thereby influencing how people perceive these models as representations of reality.

A critical situation encompasses more than mathematical considerations and their implications. Such circumstances yield concrete outcomes that directly affect individuals, demonstrating the intersection of abstract mathematical concepts and real-world consequences.

Incorporating a socio-ecological context into mathematics education means bringing to the fore critical situations where the abstractions of mathematics and challenging conditions and phenomena such as climate change, pandemics, sustainability, and humans' impact on non-human nature meet in inseparable ways (e.g., Coles et al., 2023b). In essence, it does not involve referencing an irrelevant or unincorporated context for mathematical activity. Rather, it introduces a meaningful situation or context that is significant for socio-ecological reasons and should therefore become an integral component of students' algebraic engagement.

Moreover, Coles (2023b) discusses how socio-political and ecological concerns are related: "When using the term 'socio-ecological', I aim to highlight that every concern about the planet's ecology has a socio-political aspect, and vice versa, every socio-political concern has an ecological aspect" (Coles, 2023b p. 17). Coles' view of socio-ecological implies that when relating mathematics education to socio-ecological contexts, one is invited to think about how the complex network of entanglements among ecological, social, and political concerns can be thought as a fundamental and constitutive part of the critical situations mobilized in mathematics education.

Accordingly, different aspects or dimensions of the socio-ecological and their interrelationships can also be expressed through mathematical relationships. For example, in a critical situation like the one above on foreclosure, a formula could include values which are not usually used to calculate mortgages and risks, such as the stability cost for a family to lose their home, or the stress and damage caused to children by a foreclosure. Also, a socio-ecological context poses challenges to establishing connections between mathematics in action and the exercise of power with respect to socio-ecological issues. For example, in Sweden the political decision to waive a CO₂ tax on air tickets is a critical situation where mathematics in action clearly connects politics with the concern for more environmental friendly forms of transportation.

In summary, within the socio-ecological context, the notion of *critical situations* has two related meanings. One concerns how students encounter socio-ecological situations that require urgent

attention. In other words, it refers to a situation that is critical for our world. The other meaning refers to features of mathematical realized abstractions that can become real and, therefore, can influence people's lives, actions, and choices. In this sense, these abstract structures of contextual character fall within the scope of students' work.

Mathematics education for citizenship in students' life-worlds

LAT sees mathematics education as critical—in the sense of important and crucial—for the development of higher forms of mathematical thinking and understanding. In contrast, CME goes a step beyond proposing a stance on the purpose of mathematics education in students' lives. Modern, high technological societies are based on systems where realized and real abstractions are part of the political decisions that affect their lives. Therefore, teaching and learning mathematics are not just a matter of being able to users of technologies, or of a mathematically qualified work force, or of the expert producers of technologies. Mathematics education can also serve the purpose of enabling students to evaluate and take a critical distance and stance toward actions and decisions based on mathematical models. In other words, citizens not only need to understand mathematics as objects in themselves, but also need to have clarity on its performative nature and on their formatting power. This is the core of the democratic competence in active citizenship in our societies (Skovsmose, 1994). In this sense, mathematics education can also be part of becoming *critical citizens*. That is, mathematics education is critical for *citizenship* in students' life-worlds.

The relationship between citizenship and mathematics concerns understanding the role of mathematics in society—for example, how it influences social structures, power relations, and the dynamics of social inclusion and exclusion (Skovsmose, 2023a). Additionally, mathematics is both a tool and an object of critical reflection—for instance, becoming aware of mathematics in action and

in context by recognizing that mathematical concepts and models represent a specific reality rather than the reality itself.

Mathematics is brought into action in life-worlds (Skovsmose, 2023a); it does not exist in isolation; it is applied and embedded within people's lived experiences within the real world in social contexts:

“[C]omplex social constructions, reflecting economic, political, religious, cultural, and discursive structures. No life-world exists a priori to such structuration. Furthermore, life-worlds become re-configured through personal beliefs, conceptions, misconceptions, and idiosyncrasies” (Skovsmose, 2023a, p. 70).

Skovsmose specifies that the construction of our life-worlds, including those of students, is shaped by different structures such as economic, political, religious, and cultural. These frameworks are subsequently reinterpreted as individual beliefs, perceptions, misconceptions, and distinctive characteristics. Straehler-Pohl (2017) offers an insightful analysis, arguing that mathematizing and de-mathematizing should be viewed not as direct opposites, but as interrelated components of a unified process. This perspective highlights the ongoing interaction between the visibility and invisibility of mathematics in practical settings and across society. When mathematics is made explicit through its applications—mathematization—it underscores its significance and utility. In contrast, when mathematical structures become less apparent due to processes such as automation or abstraction—de-mathematization—they continue to exert substantial influence, albeit often unnoticed. The critical and ethical concerns are twofold: the influence of algorithms on our understanding of the world, and the potential for this influence to be problematic. These concerns are highly connected to the abstract ideas and tools of algebra, such as mathematical models as algorithms, and therefore these concerns are relevant to school algebra and to letting students also develop a critical understanding of the use of algebra in society.

A need to explore new understandings of early algebra in practice

Expanding the theoretical assumptions outlined above influences both teaching practices and approaches to early algebra. Incorporating CME leads to a view of early algebra as being situated within students' life-worlds, reflecting a complex interplay of ecological, social, political, and ethical dimensions, rather than as objective or neutral abstractions. This also involves how powerful early algebraic ideas, culturally and sociologically speaking, can be integrated into these practices. This recognition carries several implications for educational practice.

These practices also need to take into consideration that students need to experience early algebra as both a form of abstraction that sets particular ways of seeing structures and relationships, and a means of unveiling abstract structures and relationships in socio-ecological contexts. Moreover, it is important to provide students with opportunities to reflect on how algebra can be applied in real-world contexts and to examine its influence on both our understanding of the world and the world itself. These practices also need to encourage students to consider how various stakeholders, including politicians, corporations, and social media platforms, may use the power of algebra to respond to critical situations such as socio-ecological crises or societal challenges for their own objectives and advantages. Furthermore, how the application of algebra can shape public perception and influence responses to such crises.

These scenarios present significant challenges for both educational practitioners and researchers, prompting consideration of how students may be engaged in activities that integrate early algebra with the previously discussed elements. In response, I held it essential to investigate new perspectives on early algebraic practices as they intersect with issues of citizenship and socio-ecological crises. This led to the formulation of an overarching research question: What characterizes activities that integrate both algebraic and socio-ecological aspects? In light of this context, an empirical study was undertaken to advance understanding of activities that

facilitate such integration for students. The subsequent chapter will provide a detailed account of this research.