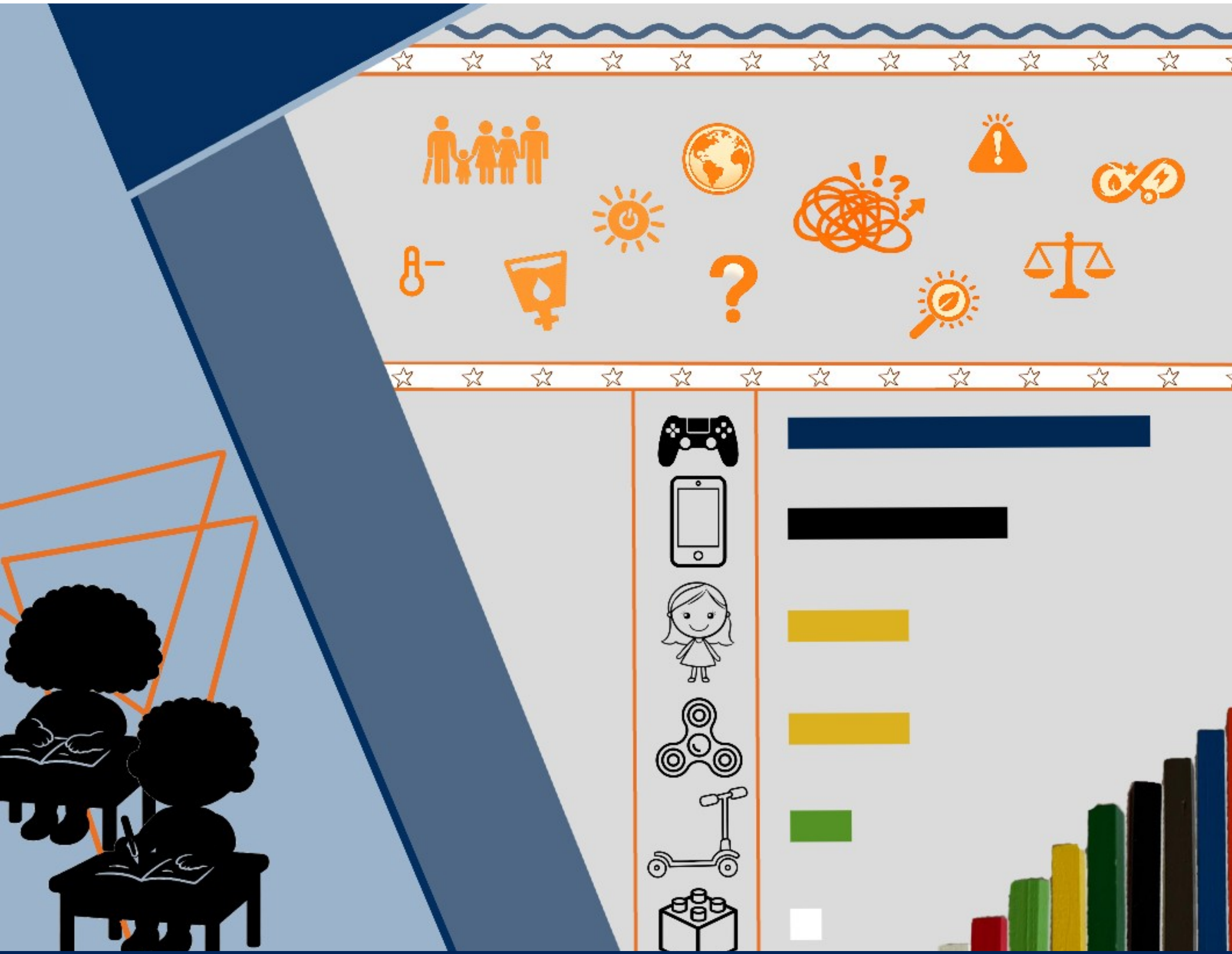
Stockholm University

The Wicked World of Algebra

Re-imagining the practices of teaching and learning early algebra in the socio-ecological

Jenny Fred



Doctoral Thesis in Mathematics Education at Stockholm University, Sweden 2026

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Jenny Fred

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics Education at Stockholm University to be publicly defended on Friday 5 June 2026 at 13.00 in Hörsal 9, hus D, Universitetsvägen 10 D and online via Zoom, public link is available at the department website.

Abstract

This thesis reimagines the teaching and learning of early algebra for students in grades K-3 in light of current socio-ecological crises. While early algebra has primarily been justified as preparation for later mathematical studies, I argue that algebraic thinking is also a key component of critical citizenship: it can help young students recognize, question, and act on the relationships, models, and values that shape their life-worlds.

The overarching aim is to examine how to create conditions that enable students in the early grades to participate in algebraic activity, first focusing on algebraic development and then expanding to citizenship within a socio-ecological context. Across three phases of work, the thesis brings together Learning Activity Theory (LAT) and Critical Mathematics Education (CME) and uses them in complementary ways. This theoretical combination broadens prevailing definitions of algebraic activity and reveals its educational possibilities.

Part 1 (my licentiate thesis) explores aspects of teaching that create conditions for students in grades K–3 to engage in algebraic work. Part 2 critically examines influential overview literature in early algebra to analyze which "powerful algebraic ideas" are foregrounded, revealing a predominance of logical and psychological framings and a relative scarcity of cultural and sociological framings connected to citizenship. Part 3 explores critical problem situations for early algebra within the socio-ecological. In this third part, a research team consisting of in-service teachers, pre-service teachers, teacher educators, and researchers has been developing the concept of Algebraic Wicked Problems (AWPs) to integrate early algebra and the socio-ecological into students' work.

The main contribution of this thesis is a conceptualization of AWPs as five interrelated features. Another contribution highlights the powerful complementarity between LAT and CME. These results suggest directions for curricula and pedagogy that position early algebra as both a form of mathematical development and preparation for critical citizenship in uncertain socio-ecological times.

Keywords: *algebraic wicked problems, early algebra, critical mathematics education, learning activity theory, socio-ecological.*

Stockholm 2026
http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-254253

ISBN 978-91-8107-622-6
ISBN 978-91-8107-623-3

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ISBN print 978-91-8107-622-6
ISBN PDF 978-91-8107-623-3

Printed in Sweden by Universitetsservice US-AB, Stockholm 2026

Till Anders, Elvira och Elton

Tack

Att skriva en avhandling kan se ut som ett ensamarbete, men i själva verket blir den till i lager: i frågor som prövats, i formuleringar som slipats, i tankar som fallit och byggts upp på nytt. Den här texten har vuxit fram i samtal och motstånd, i seminarierum och vid kaffemaskinen, och i de mellanrum där någon annans tilltro fått bära när min egen sviktat. Utan det stöd, den vägledning och möten längs vägen hade denna avhandling varken kunnat påbörjas eller slutföras.

Mycket av det som burit arbetet syns inte i texten: ett uppmuntrande ord i rätt stund, en skarp kommentar i marginalen, en fråga som öppnat ett nytt rum när jag kört fast.

Det finns många jag vill tacka—ni som har lyssnat när det behövts, frågat när det varit för lätt att nöja sig och hållit mig kvar vid det viktiga när tröttheten tagit över. Den här avhandlingen är skriven av mig, men den bär spår av många händer.

Först och främst vill jag rikta ett varmt tack till mina handledare, Hendrik van Steenbrugge och Paola Valero. Med skärpa, uthållighet och generositet har ni utmanat mitt tänkande, hållit riktningen stadig när jag själv tvekat och hjälpt mig att steg för steg mejsla fram detta arbete. Era insikter, er klokskap, er omtanke och er förmåga att både uppmuntra och pröva mina resonemang har varit avgörande för avhandlingens utveckling.

Tack även till Inger Eriksson och Lisa Björklund-Boistrup, som var mina handledare under licentiatuppsatsen. Med er skicklighet, klokskap och värme lade ni grunden till den första delen av den här avhandlingen och gjorde det möjligt för mig att ta nästa steg.

Jag vill också tacka kollegor och forskningsmiljöer, SoCAME och forskarskolan RelMaS, som på olika sätt har burit arbetet framåt med perspektiv, kritiska frågor och uppmuntran. Ett extra tack till Eva Norén och Anna Pansell för noggrann läsning av mitt arbete i slutskedet och för vägledning som gjorde

skillnad när det som mest behövdes. Tack även till Kajsa Bråting, som tyvärr inte längre finns med oss, och till Lisa Österling, som var mina 50 % läsare och vars utmanande frågor hjälpte mig att hitta vidare. Tack till Alf Coles för vårt inspirerande samtal vid mitt 90 % seminarium och för den bekräftelse som gav mig mod att fortsätta—som en stilla påminnelse om att jag faktiskt var något viktigt på spåren.
Ett särskilt tack riktas till er lärare och lärarstudenter som deltog i det kollaborativa utforskandet. Er generositet, öppenhet och vilja att dela med er av era erfarenheter har gett arbetet dess förankring och riktning—utan er hade detta inte varit möjligt.
Ett särskilt tack till Sanna som i slutfasen hjälpte mig med referenslistan. Så tacksam för att få låna dina starka och kritiska ögon.
Tack Anette, Anna-Karin.och Laura för att ni varit mina hjältar och stöttepelare när det har varit som tuffast och för alla skratt, upptåg och samtal. Och tack också till dig Nhu för din omtanke och stöttning.
Tack, Risottoklubben—Anna-Karin, Inger, Martin och Sanna—för gemenskapen, för samtal som lossat knutar, för skratt som lättat på trycket, och för ert stöd, både i det lilla och i det stora.
Tack också till Learning study-gänget för att ni funnits där för mig i både toppar och dalar som skrivandet av en avhandling innebär.
Jag vill också tacka alla mina fina vänner Christine, Erika, Lotta, Anneli, Lisa, Åsa och mina kära syskon, Pelle och Jens. Tack för att ni har funnits som ett slags fast mark när arbetet och självkänslan har gungat—för att ni har orkat lyssna, svarat med hejarop och kärlek, och burit in lätthet när jag själv blivit för tung av text. Ni har hållit dörren öppen till livet bortom avhandlingen, och just därför också gjort det möjligt för mig att ta den i mål.
Mamma och pappa i himmelen—er kärlek har funnits inom mig. Jag tänker att ni finns där ovan molnen och ser ner på mig med stolthet och värme. Och pappa: jag hör dig, som alltid, skryta högljutt om mig—som om orden kunde bära hela vägen fram.
Slutligen, och viktigast: tack, älskade familj—Anders, Elvira och Elton. Ni är min trygghet och min hamn; hos er finns kärlek, glädje, ro och vila. Elvira, tack för din förmåga att peppa, dina kramar och för din tilltro till att jag fixar det här—och ett extra

tack för att du har skapat bilden till avhandlingens framsida. Elton, tack för att du bara funnits där omkring mig, nära och självklar, och för dina kramar när jag behövde dem som mest. Anders, jag vore inget utan dig. Utan ditt stöd och din kärlek hade det inte blivit någon avhandling. Tack för att ni, med tålamod och förståelse, följt mig genom allt det som inte syns i texten men som gjort den möjlig. I love you to the moon and back!

Denna avhandling är skriven av mig—men den har vuxit fram i omsorg, tålamod, frågor och närvaro, och den bär spår av många händer. Och utan all kärlek, stöd och uppmuntran är jag tveksam att det hade blivit någon avhandling.

Glömsta, april 2026

List of articles

1. Fred, J., & Boistrup, L. B. (2017). Students' qualitatively different ways of experiencing pattern generalization. *Quaderni di Ricerca in Didattica (Mathematics), 27*(2), 155–162.
2. Fred, J. (2019). Att designa för elevers deltagande i ett algebraiskt arbete: Elever i årskurs 2 och 3 utforskar visuellt växande mönster. *Nordisk matematikkdidaktikk NOMAD[Nordic Studies in Mathematics Education]*, *24*(3–4), 107–130.
3. Fred, J., Valero, P., & Van Steenbrugge, H. (2022). Powerful algebraic ideas: Early algebraic thinking and active citizenship. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12),* (pp. 489–496). Free University of Bozen-Bolzano and ERME.
4. Fred, J., Valero, P., &, Van Steenbrugge, H. (2023). Algebraic wicked problem: Bridging the gap between early school algebra and socially relevant issues? In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13).* Alfréd Rényi Institute of Mathematics & Eötvös Loránd University of Budapest.
5. Fred, J., Van Steenbrugge, H., & Valero, P. (2024a). Connecting early algebra and sustainability through dilemmas in Algebraic Wicked Problems. In Twohill, A., Isler-Baykal, I., Knuth, E., Ribeiro, A. J., & Miller, J. *Proceedings of the 15th International Congress on Mathematics Education (ICME15), Topic Study Group 1.2: Teaching and learning of early algebra,* (pp. 16–23). Sydney, Australia.
6. Fred, J., Valero, P., & Van Steenbrugge, H. (2025). Integrating the socio-ecological and early algebra. In K. le Roux, A. Coles, A. Solares-Rojas, A. Bose, C. P. Vistro-Yu, P. Valero, N. Sinclair, M. Makramalla, R. Gutiérrez, V. Geiger, & M. Borba (Eds.), *Proceedings of the 27th ICMI Study Conference Mathematics Education and the Socio-Ecological* (pp. 464–471). MATHTED — ICMI.

Contents

Preface

With nearly a decade of experience challenging the teaching and learning of students in grades K-3, I am committed to helping learners appreciate the importance of engaging deeply with challenging problems—not only those related to algebra, but also those that equip them for the complexities of our current world and an uncertain future. My overarching goal as a teacher and teacher educator working in different research projects has been to foster curiosity and enthusiasm for mathematics and global understanding, enabling students to imagine things that they do not yet know exist. For individuals, such as myself, who have spent many years as schoolteachers, the decision to pursue a career in research can represent a significant transition. My research trajectory has progressed through distinct phases. From 2015 to 2019, I completed my licentiate studies[1] in collaboration with the National Research School of Learning. Subsequently, between 2020 and 2026, I pursued my doctoral education within the graduate program "Relevancing Mathematics and Science Education" (VR 2019-03679). In my licentiate thesis (Fred, 2019b), I identified didactical features for establishing and sustaining algebraic work among younger students. In the subsequent stage of my doctoral studies, I have adopted a Critical Mathematics Education approach (e.g.,Skovsmose, 1994), exploring how teaching can be designed

[1] In the Swedish educational system, a licenciate degree is a postgraduate research qualification between a master's and a PhD degree, often serving as a "half way" milestone towards a PhD. In the field of mathematics education, the licenciate has been a way for many teachers to acquire research competence to support a research-based development of teaching and learning in schools.

to enable students to integrate algebraic thinking with socially relevant issues in their learning process.

In this thesis, I bring together two fields that have not usually been close to each other—Early Algebra and Critical Mathematics Education—and I pose challenges to both. I am challenging the socio-political research community to recognize algebraic thinking as a fundamental component of school mathematics for citizenship. Simultaneously, I encourage the research field of Early Algebra to consider socially relevant issues as integral aspects of algebraic activity, rather than merely as contexts for word problems. When presenting my work within early algebra communities, I am often asked, "Where is the algebra in your research?" Conversely, audiences in the socio-political research and Critical Mathematics Education inquire, "Where are the socio-ecological issues in your research?" Nonetheless, scholars from both domains acknowledge the significance of my efforts to bridge these typically disconnected areas. Throughout this thesis, I will describe various movements and developments when bringing these fields together. This PhD thesis marks the initial step in my scholarly journey, aiming to investigate how early algebraic aspects and socially relevant concerns can be meaningfully integrated.

The research journey since I started the licenciate in 2015 may be likened to venturing forth upon an uncharted sea, where the direction of the wind is unpredictable and the navigational tools at one's disposal are not predetermined. The wind's strength and direction vary, as do the resources available for guidance. The process of research as it unfolded has been fraught with uncertainties and exploration. Constructing an intelligible narrative "a posteriori" for the whole process has been a challenging task wherein the insights gained from prior research served to fuel the evolution of my concepts, as they were shaped by new texts I studied and the discussions I engaged in with a wide range of colleagues and collaborators. My ideas expanded, moved, and transformed, reflecting the dynamic interplay between researching and dialogue during almost ten years of working with the passion of developing early algebraic teaching practices that matter for children, teachers, and schools.

Thus, in writing this manuscript that reports on the whole research journey from 2015 until now, I have devoted considerable time and attention to developing a narrative that describes the research undertaken and the insights and results gained along the way. I have approached this writing with the understanding that it represents only one stage in an ongoing process, rather than its culmination. My intention has been to provide a comprehensive description of my doctoral research work, and also set a foundation for further exploration and development.

Chapter 1 – Approaching the research problem: Early algebra in current socio-ecological crises

Children's engagement with and mastery of school algebra are important moments in school mathematics. While there seems not to be a general agreement on how algebra teaching and learning are to be conducted (Kieran, 2020, p. 36), there is agreement on school algebra being "a way of expressing generality; a study of symbol manipulation and equation solving; a study of functions; a way to solve certain classes of problems; and a way to model real situations" (Stacey & Chick, 2004, p. 16). The teaching of algebra aims to foster students' capacity of "expressing generalizations, establishing relationships, solving problems, exploring properties, proving theorems, and calculating" (Arcavi et al., 2017, pp. 2–3). Algebra holds a particular position within mathematics education, due to its applicability across other areas of mathematics and its support for general reasoning, drawing conclusions, and constructing proofs (Cai & Knuth, 2005). Therefore, students' knowledge of algebra can enhance their understanding of other areas of mathematics. For this reason, algebra has also been described as a major gateway to later mathematics (Kaput, 2008).

Despite its centrality and importance, developing students' understanding of algebraic concepts is a challenging task in classroom practice, and it also poses problems for research (Kaput et al., 2008). In Sweden, various reports indicate that students at different stages of schooling have difficulties with learning algebra (Bråting, 2023). For example, results of the Trends in International Mathematics and Science Study (TIMSS) have shown that Swedish students perform below the international average in algebra (Madej, 2022). Another report in this direction addresses how, although overall mathematics results improved slightly in 2015, performance in algebra remained comparatively low (Vetenskapsrådet, 2017). To counteract such trends, there have been various initiatives targeting algebra, such as professional development for teachers, the “Algebra for All” initiative, and practice-based research projects funded by the Swedish Institute for Educational Research—such as “Developing Algebraic Reasoning Capability” (e.g., Eriksson et al., 2019, 2021, 2024) led by Professor Emeritus Inger Eriksson of Stockholm University. Despite these concerted efforts to improve algebra instruction and learning in schools, substantial gains in students’ algebra performance have not been observed (Bråting, 2023). In other words, neither learning nor teaching algebra is easy.

The field of Early Algebra

In recent decades, there has been a growing interest in mathematics education research and practice for understanding children’s algebraic thinking and reasoning in lower grades (Kaput, 2008), and in how to teach promoting such forms of thinking in these grades (Cai & Knuth, 2011). It has been shown that conjecturing about relationships (Coles & Ahn, 2022), introducing equations (Radford, 2022) and reasoning about structural aspects of algebraic expressions (Wettergren, 2022) can be powerful routes to promote young children’s engagement with algebra. In most of these cases, the relevance of the research has been argued in relation to how algebra teaching and learning can affect students’ continued learning in mathematics (Carraher, Schliemann, et al., 2008). In response to these kinds of insights and questions, scholars in mathematics education research established a specialized field known as *early algebra* (e.g., Cai & Knuth, 2011; Kaput et al., 2008; Kieran, 2018a; Kieran et al., 2016). Researchers in this new field emphasize that the teaching and learning of early algebra do not focus on formal symbolic manipulation. Instead, the focus became to problematize algebraic thinking related to identifying and expressing relationships, patterns, structures, and generalizations. Their interests have also included the use of representations, instructional design in classroom practices, the role of the teacher, and the development of theories of how early algebraic thinking develops (e.g., Kieran, 2022; Kieran et al., 2016).

The above addresses, as I see it, three central points. First, teaching and learning early algebra differ from practices in advanced grades of education. Second, the teaching and learning of early algebra so far is mainly justified with respect to development of algebraic thinking needed for children's further mathematical development. And third, the field of early algebra research and practice is still under construction, so new insights can still emerge (Kieran, 2022). There is no doubt about the importance of supporting students' early and further algebraic and general mathematical development. However, it is also important to recognize the broader issue of how mathematical knowledge gained in educational settings prepares all students—not only those who continue with advanced mathematics—for their engagement with the world in which they live and their future roles within society as active citizens.

Early algebra for citizenship

Addressing the role of mathematics education in general, and algebra in particular, for citizenship is not new. For example, in the United States, the work of civil rights activist and educator Bob Moses in the "The Algebra Project Inc." (n.d.) highlighted the importance of access to school algebra in current technological-driven societies for increasing educational and economic opportunities for traditionally marginalized and systemically impoverished Black people (Moses & Cobb, 2001). Moses made clear that understanding and mastering algebra opened the possibility of accessing the symbolic language at the basis of current technological development. Therefore, being counted as a citizen in such society demands people to have a mathematical literacy—highly dependent on algebra—to understand and participate in different social and work organization processes. This argument clearly sets the political dimension of algebraic thinking and algebra learning for citizenship.

Similarly, in Scandinavia, Ole Skovsmose (e.g., Skovsmose, 1994, 2023a), proposed his ideas of Critical Mathematics Education (hereafter referred to as CME). Skovsmose took a somewhat different approach from Moses. While both addressed issues such as social exclusion, the pursuit of social justice, and the need to expand opportunities for students, Skovsmose placed particular emphasis on the necessity of engaging with mathematics *critically* in all its forms and applications (e.g., Skovsmose, 2014). He illustrated this by, for example, highlighting the often "invisible" power of mathematics, noting that it shapes our perceptions and influences the way we construct our world (Skovsmose, 2023a). Skovsmose considered mathematics—especially through modelling—as a discipline capable of offering both benefits and drawbacks, thereby establishing its inherent connection to uncertainty and the values underlying its use (Skovsmose, 2021). His argument stressed that critical thinking as well as a critical stance toward the uses of mathematics constitute an essential component of mathematical literacy, especially within STEM-driven societies where technology and crises are closely interrelated (Skovsmose, 2023a). This critical perspective is of importance for all students, not only those pursuing advanced studies or careers in mathematics. It is possible to conclude that, while learning algebra has intrinsic value for developing key mathematical skills, its broader impact extends to personal growth and active participation in society. The issue remains of how to think about what early algebra, its teaching and learning, may entail with respect to the current world in which children now and as full citizens in the future live.

Early algebra in the socio-ecological

It is the year 2026, and the world is facing various socio-ecological crises, a term that refers to the intertwined nature of the multiple exacerbated social and ecological predicaments of this time (Vestergren et al., 2024). These crises include climate change, sustainability, the aftermath of the COVID-19 pandemic, food security, increased inequalities, migration, and the human destructive impact on the nonhuman world. As citizens, politicians, parents, and children, we face many dimensions of these crises; and each one brings environmental, social, societal, and ethical implications and poses dilemmas. Media such as newspapers, radio, television, and diverse social media platforms are responsible for broadcasting these reports. In Sweden, there is a prevalence of reports, such as OECD Environmental Performance Reviews: Sweden 2025 (OECD, 2025), on the following environmental issues: rising sea and ocean levels, water and air pollution, global warming, climate change, food waste, biodiversity loss, ocean and lake acidification, and the general destruction of the Earth's natural resources. These frequent reports serve a dual purpose: they inform the public and inspire many to reconsider their habits and adopt a more sustainable lifestyle. As members of the global community, we are urged to take responsibility for these socio-ecological crises by making sustainable choices in our daily lives.

In the context of our day-to-day activities, we also face socio-ecological dilemmas. It can thus be posited that the ethical values of any individual are subject to close examination when faced with the daily decision-making involved in, for example, choosing between using private vehicles and opting for public transportation or cycling. At the household level, the promotion of renewable energy sources over fossil fuels as sources of energy, as well as issues of waste production and recycling, emerge as salient concerns for families. Parents and children are faced with multifaceted sustainability choices, encompassing a wide range of environmental, social, and economic factors.

As a citizen of a globalized world, I have begun to question the long-term sustainability of my own lifestyle and that of the planet.

The issue under consideration at an individual level is whether I can continue making the same choices I have in the past in my roles as citizen, parent, friend, and sibling. I have been confronted with the question of how do such considerations in my personal life connect to my role as a mathematics teacher, a teacher educator and researcher in early algebra? Is it at all important to establish such connection?

In my multiple roles and particularly as a researcher and teacher, a *research problem* emerges: Can early algebra be important not only for mathematical development but also for critical citizenship in a world facing socio-ecological crises? What would it take to establish such a connection in theory and practice? Following this line, I propose that early algebra can be important not only for mathematical development but also for critical citizenship in a world facing socio-ecological crises. Therefore, this thesis sets out to bring together two dimensions of teaching and learning early algebra in the current world in which we are living: early algebra for the purpose of developing further forms of mathematical thinking and early algebra for citizenship within a socio-ecological context. That is, the field of early algebra is examined through the lens of the development of algebraic thinking, with particular attention to how this knowing relates to children's potential as participants in society within a socio-ecological context. I argue that exploring innovative approaches to teaching and learning early algebra in schools is of paramount importance. We need to develop approaches that emphasize the role of early algebra both in terms of mathematical development and for preparing students to become critical citizens in today's world with its complex socio-ecological crises.

Positioning the research problem in the research field

The research problem in this thesis can be related to developments in mathematics education research, including a shift toward researching diverse social, political, and cultural dimensions of mathematics education (Keitel, 2013). In research, this change is addressed as the socio-political turn (see Gutiérrez, 2013). Valero (2010) observed that shifts in mathematics education research can be examined through the identification of what is considered to be legitimate research objects within the field. In the following section, I will detail these shifts utilizing Valero's descriptions.

Initially, as mathematics education began to evolve into a scientific research field, *mathematics* was at the center (Kilpatrick, 2006; Silver & Herbst, 2007). The research objects were mainly concerned with problems of practice and/or the improvement of practice with a focus on mathematics. In conclusion, there was no obvious focus on the didactics. A first movement toward a didactic focus was related to a growing interest in the didactic triad (see Cobb et al., 1992; see also Valero, 2010). Studies concentrated on *students*' conceptual understanding and *teachers*' methods for teaching *mathematical concepts*. Subsequently, a second development saw heightened attention to the didactic triad within *classroom settings*, with research focusing on classroom dynamics and delineating the classroom as a distinct boundary for the relationships of the elements in the triad. In the third turn, scholars' focus expanded to consider the relationship between the didactic triad and its broader *context*, prompting research into educational practices that extend beyond the immediate parameters of the triad itself (Valero, 2010).

Currently, a new movement is underway that focuses on the intricate relationships between socio-ecological crises and mathematics education, what researchers have been recently calling the *socio-ecological turn* in mathematics education (le Roux et al., 2025). As Coles (2022) notes, this development prompts us to use the term “socio-ecological crisis” rather than simply “environmental crisis,” as it reflects an emerging perspective in mathematics education where neither social nor ecological factors are treated as fixed backgrounds for one another.

Indeed, as a response to the complexity of the current socio-ecological challenges, an international research initiative in mathematics education has been initiated: the International Commission on Mathematical Instruction’s Study 27 (ICMI Study 27), formally titled “Mathematics Education and the Socio-Ecological” (Coles et al., 2024). The purpose of this initiative is to map and analyze existing research and practice in mathematics education, and to explore possible future directions for research, policy, and teaching related to complex social and ecological issues such as climate change, inequality, marginalization, and ethical considerations.

This thesis is situated and contributes to the socio-ecological turn in mathematics education research by addressing the problem of how the teaching and learning of early algebra and socio-ecological contexts can be fundamental to student engagement with mathematical problems, where algebraic and socio-ecological aspects are interdependent and inform one another.

Aim and research questions

As argued above, scholars in early algebra research emphasize the importance of providing younger students with opportunities to develop algebraic thinking. Skills such as identifying and expressing relationships, patterns, structures, and generalizations are essential, as they can deepen students’ understanding of various mathematical domains. Furthermore, the importance of teaching that engages students in algebraic activities within relevant social contexts is emphasized. Relevant questions include how such work may be initiated and sustained in classroom practice, how teaching can be designed to support students’ development of algebraic thinking, and what implications may follow when the aims of teaching are broadened to encompass aspects of citizenship within a socio-ecological context.

Thus, this thesis proposes to investigate the simultaneous integration of socio-ecological considerations into the teaching and learning of early algebra. This study connects the intention to empower students as future citizens within the socio-ecological concerns of today with the intention to open possibilities for new curricula and pedagogies that promote algebraic thinking.

The overall aim of this thesis is to examine how conditions can be created that enable students to participate in algebraic activities in the context of early algebra. In the first part of the thesis, this is explored in relation to purposes and considerations concerning students' continued learning of mathematics. The second and third parts adopt a broader perspective, considering aspects of citizenship within a socio-ecological context.

The research is structured around three overarching questions:

1. What aspects of teaching can create conditions for students of younger ages to be involved in algebraic activities?
2. How can we understand algebraic ideas for the combined purpose of developing younger students' algebraic thinking and fostering citizenship?
3. What characterizes activities that integrate both algebraic and socio-ecological aspects?

The outline of the thesis

The thesis is structured in three parts. The first part is my licentiate thesis, which has already been examined and approved (Fred, 2019b). It addresses the first research question, and it is presented in Chapter 2. It focuses on creating conditions for younger students to engage in algebraic activities in the context of early algebra. The second part addresses the second research question and addresses a turning point toward a new terrain in early algebra, when adopting CME broadened possible conceptions of the concerns and purposes of teaching and learning algebra for citizenship. This part is unfolded in Chapter 3. The third part explores the third research question and focuses on expanding ideas for teaching early algebra in the lower grades integrating critical mathematics education and socio-ecological contexts. This part is presented in Chapter 4 that maps existing research supporting this expansion, and Chapter 5 that presents the empirical experimentation. Chapter 6 describes the findings in relation to the third research question; Chapter 7 puts forward a final discussion conclusion that brings threads together and opens further opportunities for the future.

In my attempt to make the overall narrative of the research trajectory intelligible, as mentioned in the Preface, this manuscript combines both the usual kappa or wrapping text with the published articles produced along the research process, where they become necessary for the reader to understand the research trajectory.

Table 1 in the next page summarizes the three different parts and their content.

Table 1
Overview of the different parts of the thesis

PART 1. Lic. Thesis Feb. 2016 – March 2019	**PART 2** Feb. 2021 – May 2021	**PART 3** May 2022 – Feb. 2025
Research question 1: What aspects of teaching can create conditions for students of younger ages to be involved in algebraic activities?	*Research question 2:* How can we understand algebraic ideas for the combined purpose of developing younger students' algebraic thinking and fostering citizenship?	*Research question 3:* What characterizes activities that integrate both algebraic and socio-ecological aspects?
Empirical inquiry 1: Identify critical aspects (regarding what to discern in their learning) of expressing and justifying pattern generalization algebraically. *Empirical inquiry 2:* Exploring how theoretical principles of Davydov's learning activity theory relate to the establishment and sustainability of algebraic activity, and to students' recognition of key aspects.	*Empirical inquiry 3:* Broadening the understanding of early algebraic thinking by bringing in critical citizenship.	*Empirical inquiry 4:* Imagining a new kind of problem situations, Algebraic Wicked Problems, that invite students to simultaneously integrate early algebraic thinking, citizenship, and socio-ecological concerns.
Chapter 2 Papers 1, 2	Chapter 3 Paper 3	Chapters 4, 5 Papers 4, 5, 6
Results (Chapter 6) and concluding discussion (Chapter 7)		

Chapter 2 – Teaching for engaging younger students in early algebraic activities

This chapter addresses the first overarching research question, "What aspects of teaching can create conditions for students of younger ages to be involved in algebraic activities?" which was explored in detail in my licentiate thesis (Fred, 2019b). The aim of the licentiate thesis was to explore aspects of teaching that create the conditions for students in grades K–3 to be involved in algebraic work. Situated in research on early algebra, the thesis adopted elements of Learning Activity Theory (LAT) to carry out two empirical studies following the principles of Learning studies with teachers in Swedish schools.

In this chapter, I synthesize important aspects of the licenciate thesis, such as the concerns and purposes of early algebra as a field of research, and the theoretical foundation of LAT, which informed my work. I also present the key insights gained from the licenciate thesis that informed and were further developed in my subsequent doctoral studies. I also summarize the main findings in the publications that resulted from the licenciate work and present them here as Paper 1 and Paper 2.

Concerns, purposes, and practices in early algebra research

My licenciate thesis was situated within the research field of early algebra. To explain the ideas that oriented my licenciate thesis, I provide an account of the underlying concerns and purposes of the research in the field and how the former translate into the research-based development of teaching practices.

Early algebra research generally takes as a point of departure the assumption that teaching early algebra plays a central role in students' continued learning of algebra and mathematics (e.g., Cai & Knuth, 2011). Research highlights problems that may arise in students' later algebraic learning, which can be traced back to the effects of early teaching (e.g., Kaput, 1999). Such teaching, to an excessive extent, consists of mechanical doing and focuses on teaching a number of procedures, what is considered problematic (Greer, 2008). The problem highlighted is that this type of teaching can prevent students from developing the ability to use algebra as a tool to solve mathematical problems, describe and analyze relations, and characterize and understand mathematical structures and ideas (Greer, 2008). Another problem highlighted in research is when teaching focuses excessively on numerical solutions, as there is a risk that students get stuck in procedures rather than developing abilities such as general reasoning and drawing general conclusions (Lins & Kaput, 2004). The problem can be expressed, for example, by students when solving algebra problems, who try to "insert" concrete numbers instead of engaging in general reasoning (Kieran, 2006).

As a result, research suggests that students be introduced to algebraic activities from the start. Algebraic activities are those that go beyond mastery of mathematical operations and instead focus on exploring underlying mathematical structures (Davydov, 2008). The premise is that the early teaching developing such form of algebraic activity will, over time, support the emergence of algebraic thinking, which in turn can create better conditions for students' continued algebraic learning. Early algebraic work can include students' engagement with treating indeterminate quantities—such as unknowns and variables—as if they were known (Radford, 2018). Both Radford (2018) and Mason (2018) underscore that the use of mathematical symbolic language is neither a necessary nor sufficient condition for students to develop algebraic thinking. This mode of thought can be articulated through natural language, physical objects, non-traditional symbols, or even with numbers used in a general manner (Radford, 2014). For teaching, this means that students need to be given the opportunity to develop abilities such as using algebraic representations (Carraher, Schliemann et al., 2008), reasoning algebraically (Blanton & Kaput, 2008), making algebraic generalizations (Radford, 2006), noticing structures (Kieran, 2006), drawing general conclusions (Carraher, Martinez et al., 2008), and overall expressing generalities.

For my licenciate thesis these concerns, purposes and practices provided a frame to design a research concerned with exploring how teachers in Swedish lower grades in the elementary school could collaboratively design concrete teaching situations that promoted students' sustained engagement in algebraic activities. In what follows, I will delve into the main theoretical framework adopted.

Theoretical orientation

Since my licenciate thesis was carried out at the same time (but it was not part of that project) as the collaborative research team led by Inger Eriksson and her project "Developing Algebraic Reasoning Capability", I was prone to engage with LAT, as outlined by Davydov (2008). In addition, elements of variation theory (Marton & Booth, 1997; Runesson, 2005) and Radford's definition of algebraic pattern generalizations (2006) were incorporated as foundational theoretical perspectives. Given the significant role that LAT plays throughout this thesis, it will be discussed in greater detail below.

LAT is a theoretical and didactic framework based on Vygotsky's (2001) cultural-historical perspective and Leontiev's (1978) activity theory. It is mainly developed by the Russian researchers Vasily Vasilovich Davydov (1930-1998) and Evald Daniil El'konin (1904-1984). As a theoretical framework, it introduced concepts related to algebraic work and thinking, and students' reflection. As a didactic framework, LAT established concepts such as *problem situation*, *learning model*, and *contradictions*.

According to LAT, learning is understood as a transformation of participation in practices shaped by cultural and historical contexts (Davydov, 2008). Specifically, students are encouraged to engage in theoretical work where they collaboratively reconstruct the historical development of a concept (Davydov, 1990). This approach challenges students to examine and identify the underlying principles, structures, and relationships inherent in a concept, thereby promoting deeper theoretical understanding (Davydov, 1990). For example, this approach frames learning as the exploration of the principles and internal relations of specific concepts, ultimately leading to a comprehensive grasp of theoretical ideas (Davydov, 2008).

Additionally, the concept of collective reflection plays a pivotal role in learning, as reflection is recognized as a key component in fostering students' theoretical thinking. Collective reflection involves articulating how another individual has thought and reasoned, enabling students to advance their own theoretical perspectives by evaluating and relating these viewpoints to their personal thinking processes (Zuckerman, 2004).

To support the didactic approach of inviting students to algebraic work, LAT concepts of problem situations, learning models, and contradictions are important to the design of teaching. The purpose of a *problem situation* is to engage students in theoretical work, positioning them as active participants (Repkin, 2003). Additionally, it encompasses various aspects of the target knowledge content and it needs to be sufficiently complex to prompt students to engage in theoretical work (e.g., Davydov, 1990).

A *learning model* functions as a mediating tool in the collective theoretical work by creating, reconstructing, and/or processing it (Davydov, 2008; Gorbov & Chudinova, 2000). Furthermore, it captures the theoretical aspects of the knowledge content to be learned and thus gives students access to it while working with it. A learning model can be material, iconic, or semiotic, taking the form of schematics, sketches, routes, or symbols (Davydov, 2008).

The learning model in Figure 1aims to illustrate the relationships among a, b, and c; and it is intended to enable students to explore structures and relationships in algebraic expressions through theoretical work.

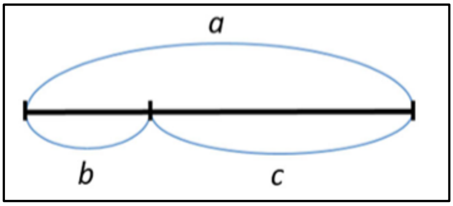

Figure 1 A learning model[2]

A *contradiction* is expressed, for example, as perceived conflicts, dilemmas, or snags (Eriksson et al., 2021). Contradictions serve as valuable tools in designing problem situations intended to prompt students' engagement in theoretical inquiry (Davydov, 2008; Zuckerman, 2003). When contradictions are introduced as hooks or similar elements, they function as catalysts for students' theoretical work (Eriksson et al., 2019). The objective in designing problem situations is to deliberately incorporate contradictions to encourage students to further investigate specific aspects of the content. For instance, contradictions might involve the teacher or a student presenting a mathematical dilemma or challenge that requires resolution, and addressing such challenges fosters students' discernment of key content components.

Methodological orientation

The purpose of the empirical work carried out in the licenciate thesis was to explore aspects of teaching that create conditions for younger students (grades 2 and 3) to be engaged in algebraic work. Learning study (e.g., Marton & Booth, 1997; Marton et al., 2004) was chosen as the research approach in the project. Learning study,

[2] The image is post-constructed from Davydov et al. (2012).

as an approach, offers a so-called "machinery" (Carlgren, 2017) for investigating and testing aspects of teaching that create conditions for younger students to engage in algebraic work. A research team consisting of two primary school teachers in mathematics and a teacher researcher (myself) worked in a collaborative and intervening process in the learning study process.

Two inquiries were conducted. Inquiry 1 was based on interviews and focused on distinguishing students' qualitatively different ways of seeing pattern generalization. The purpose was to identify critical aspects[3] (regarding what to discern in their learning) of expressing and justifying pattern generalization algebraically. Inquiry 2 was an intervention study that focused on the functions of the theoretical principles of learning activities for the establishment and maintenance of algebraic work. The empirical data for both studies consisted of (1) lesson plans of the designed lessons; (2) video recordings of three research lessons; (3) a synopsis of video-recorded research lessons; (4) transcriptions of parts of research lessons; and (5) videotaped interviews with eight students and transcriptions thereof. The results of the empirical inquiries were published in Paper 1 and Paper 2, respectively.

Findings

I synthesize here the results in relation to the licentiate thesis' overarching question: What aspects of teaching can create conditions for students of younger ages to be involved in algebraic activities? This question was addressed by two empirical enquiries, each addressing the two following research questions, respectively:

[3] Critical aspects constitute the essential components of a mathematical concept or phenomenon, such as a number pattern, that learners must recognise to develop a comprehensive understanding. Failure to address these aspects may lead to incomplete or inaccurate comprehension. Within variation theory, a critical aspect is an element vital to accurately grasping a particular phenomenon or concept.

1. What aspects do students need to be given the opportunity to distinguish in order to be able to express and argue for a pattern generalization algebraically?
2. What functions can the learning activity theory principles have in establishing and maintaining algebraic work?

I summarize the findings of the two empirical enquiries in the two sections below. Together, these findings provide clear indications of how to understand students' discernments of algebraic structures in teaching, of the problem situations we present to students, and the different dimensions that play a central role in establishing and maintaining students' algebraic work.

Empirical inquiry 1. Discerning algebraic structures

The findings of this empirical inquiry help to understand what discernment of the algebraic structures of patterns can mean. The student actions that indicate such discernment could be seen in relation to the identified critical aspects of expressing pattern generalizations algebraically. Thus, there seems to be a certain connection between which algebraic structures students pay attention to and their ability to express and argue for pattern generalizations algebraically. Thus, when students discern the critical relationship between an element's position and the number of components, this can be seen as an example of having discerned the element's internal algebraic structure.

Radford (2006) emphasizes that pattern generalizations involve comparing the mathematical aspects of patterns and then generalizing from those comparisons. Furthermore, Rivera (2013) describes it as the ability to relate the mathematical aspects of individual elements to one another. Based on the analysis in Paper 1, the value of being able to compare the positions of specific elements with the number of components, and of being able to distinguish that the same relationship applied to all elements in the pattern, emerged. More specifically, it was about the students expressing that the element consisted of as many columns of squares as its position indicated. Here, for example, the student could describe it as making one column at element 1, two columns at element 2, and so on.

To understand the validity of the critical aspects in relation to students' emerging algebraic thinking, these aspects can be discussed in light of Kieran's (2018b) description of algebraic thinking. In relation to Kieran's description, the discernment of critical aspects could be seen as the ability to see through the pattern's structure, break it down into components, and create new structures. This was expressed, for example, in the students' actions in both Paper 1 and 2, when they seemed to highlight the critical aspect of the relationship between the figure's position and the number of elements. The students broke down the pattern into structures that enabled comparison of the internal algebraic structures of specific elements. This also happened when the students were asked to discern the critical aspect of using the relationship between an element's position and the number of components to predict an arbitrary element in the pattern. Regarding this critical aspect, it emerged that students also used internal algebraic structures to predict an arbitrary element (Kieran, 2018b). The result can thus provide insight into how students' discernment of the identified critical aspects can create conditions for the emergence of algebraic thinking.

The results in Paper 1 regarding the critical aspects can also be discussed in relation to what Blanton and Kaput (2008) and Carraher and Schliemann (2014) describe as functional relations, which are relations between two quantities where one quantity affects the other. The critical aspects thus describe the functional relationship between an element's position in the pattern and the number of components, i.e., the number of components in an element is a function of its position.

Empirical inquiry 2. Establishing and maintaining an algebraic work

The didactic principles of LAT: problem situation, learning model and contradiction have on an overall level had functions such as 1) directing the students' attention to the critical aspects, 2) as tools in the students' arguments, 3) stimulating collective exploration and reflective processes.

In LAT, it is emphasized that the problem situations students face are crucial for establishing algebraic work (Davydov, 2008). The analysis in Paper 2 shows that a complex interplay among tasks, students' actions, and the teacher's orchestration of students' work creates the conditions for a problem situation to be realized. The result thus strengthens Lindberg's (2010) descriptions of how the realization of teaching is shaped by a complex interplay between tasks, instructions, the orchestration of students' work, knowledge content, students, and teachers.

Ryve et al. (2013) highlight how mediating tools can increase the conditions for establishing productive classroom discussions. In LAT, the idea of learning models is highlighted as a tool for materializing students' collective exploration, in which exploration occurs through students creating, reconstructing, and/or processing the models (Gorbov & Chudinova, 2000). The results in Paper 2 show that the learning model, as a mediating tool, played a central role in both the establishment and maintenance of algebraic work. In the analysis, two main functions emerged that the learning model had: 1) visualization of critical aspects, 2) tools for classroom communication and reflective discussions. Visualization of critical aspects can be discussed in light of Eriksson et al. (2019, 2024), who emphasize the learning model's function in materializing students' arguments, thereby making the specific meanings of those arguments accessible to students. In Paper 2, the learning model 1) visualized the relationship between the position of the element and the number of components by taking the form of groupings of components in given elements, and 2) served as a tool for classroom communication and the reflective discussions regarding the pattern's algebraic structures. An additional factor that seemed to contribute to the learning model's function as a tool in classroom communication and the reflective discussions was that it was staged, established, and transformed in front of the common board. In other words, it seemed to contribute to it also becoming a tool for other students, who did not verbally participate in the discussions, to distinguish the algebraic structures of the pattern.

Coles and Brown (2016) emphasize that, when planning teaching, teachers focus on the how of directing students' attention to aspects of the content. Contradictions can be discussed in terms of how the teaching can direct students' attention to critical aspects. Eriksson et al. (2019, 2024) describe it as building contradictions into the problem situations, thereby creating a need for students to explore the aspects of the content. In relation to the results in Paper 2, the contradictions can be seen as directing students' attention to critical aspects.

Another key factor that emerged in the analysis was to build the critical aspects into the instructions and questions related to the staging of the problem situation, to challenge students to explore the algebraic structures of a pattern (Larsson, 2015). For example, the result in Paper 2 shows that the teacher or assignment did not explicitly ask the students to compare the relationship between an element's position and the number of components and then use that relationship to predict a non-given element. The previous point can also be discussed in light of Larsson's (2015) idea that the teacher does not give direct instructions on what students should do, but rather implicitly challenges them to discern content aspects.

Blanton and Kaput (2011), Larsson (2015), and Warren (2005) particularly emphasize the teacher's actions, specifically their ability to identify, act on, and exploit mathematical situations that arise in teaching. Paper 2 highlighted the value of both the teacher's and the students' actions in establishing and maintaining the algebraic work. In other words, it was not only about the teacher's actions but also about a cooperative dynamic in which the teacher based his actions on the students' actions, or vice versa. More specifically, it was about the teacher exploiting students' participation in the various situations that arose during classroom work and/or creating new problem situations to further challenge students to explore the algebraic structures of patterns. It thus appeared that the teacher's actions and the students' actions were being pushed in the same direction.

I

Paper 1: Students' qualitatively different ways of experiencing pattern generalization

Fred, J., & Boistrup, L. B. (2017). Students' qualitatively different ways of experiencing pattern generalization. *Quaderni di Ricerca in Didattica (Mathematics),* 27(2), 155–162

This article examined the experiences of second- and third-grade students with pattern generalizations, focusing on critical aspects related to their ability to express, justify, and argue for a pattern generalization algebraically. The following research question was addressed: "What aspects do students need to be given the opportunity to distinguish in order to be able to express and argue for a pattern generalization algebraically?" Employing variation theory (Marton & Booth, 1997; Marton et al., 2004) as an analytical framework, four qualitatively distinct categories emerged regarding students' engagement with pattern generalizations.

Category 1 refers to the experience of pattern generalizations as groupings based on identifiable structures that are not necessarily mathematical in nature. The students' groupings primarily served as descriptive statements rather than as predictive tools for determining the number of components in a specific element.

Category 2 refers to the recognition of pattern generalizations as additive structures, emphasizing adjacent elements and quantifying the increase in components—such as squares or matches—that constitute each progression. Students determined the differences between the quantities of components in sequential elements and verified consistency across all provided examples.

Category 3 characterizes the experience of pattern generalizations as one-dimensional relational structures, emphasizing the correlation between the number of components and the position of each element. The analysis highlights only the internal structure of individual elements, specifically focusing on the relationship between each element's position and its corresponding number of components.

Category 4 characterizes the experience of pattern generalizations as two-dimensional structures, with a focus on both internal and external structural relationships among elements. Specifically, attention was directed to the interplay between the position of individual elements and the quantity of components, as well as the ways in which relationships among the 56 distinct elements could be interlinked. Additionally, the concept of a distinct relationship was applied to signify an arbitrary element.

Based on the categorization above, three critical aspects were identified regarding students' ability to express and argue for a pattern generalization algebraically: (a) to discern the relationship between the position of elements and the number of components; (b) to discern how to use the relationship between the position of an element and the number of constituents to predict an arbitrary element of the pattern; (c) to distinguish the constant (the component that does not change but remains the same in elements) of the pattern.

"Quaderni di Ricerca in Didattica (Mathematics)", n. 27, Supplemento n.2, 2017
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Expressing and justifying pattern generalization algebraically

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Abstract. The main objective in this paper is on learning more about younger students' emergence of the ability to express and justify pattern generalization algebraically, particularly in relation to what aspects students need to discern to be able to express and justify pattern generalization algebraically. This forms a point of departure for discussing the meaning of making algebraic generalizations in the early grades. The findings constitute a foundation for a project on classroom teaching and learning in mathematics, carried out as a collaboration between researchers and teachers.

Résumé. L'objectif principal dans ce document est d'apprendre plus sur les façons d'expérimenter la généralisation des schémas par les élèves plus jeunes et ce à propos de quels aspects les étudiants doivent discerner pour pouvoir exprimer et justifier les généralisations de formes algébriquement. Ceci comme un point de départ dans les discussions concernant la signification de faire des généralisations algébriques dans les premières années. Les résultats constituent une base pour un projet de développement de cours en collaboration entre chercheurs et enseignants.

1. Background

Researchers (e.g., Greer, 2008; Usiskin, 1988) advocate alternative approaches to the teaching of algebra, since the literature reveals that teaching today often focuses on learning a number of procedures rather than creating the conditions for enabling students to develop abilities such as reasoning algebraically, making algebraic generalizations, and using algebraic representations. Furthermore, Usiskin (1988) and Greer (2008) highlight how teaching which does not go beyond the practicing of manipulative skills, instead of developing understanding, can prevent students from using algebra as a powerful tool for solving mathematical problems. Included here are processes like describing and analysing relationships, characterizing and understanding mathematical structures and ideas (e.g., Davydov, 2008; Kaput, 2007; Kieran, 2006; 2004; Radford, 2010, 2014). In relation to enabling younger students to develop algebraic understanding, Mason (1996), Radford (2006), and Warren (2006) all suggest the use of mathematical patterns as an introduction.

2. Generalizations in relation to mathematical patterns

In research regarding mathematical patterns and generalizations, there are different descriptions of the meaning of making generalizations. Radford (2006), for example, highlights how generalization is about different layers of consciousness; to perceive the pattern's mathematical structure; to perceive the commonality of the pattern; to generalize a local commonality to all the parts of the sequence; as well as being able to express the general. In a more recent article, Radford (2011) stresses the ability to generalize in relation to being able to perceive both the pattern's spatial and numerical regularity, where the spatial structure is about, for example, how matches may be positioned in patterns (see figure 1). This entails distinguishing how both the numerical and the spatial structures belong together, including what is equal and what separates them, and then to abstract this commonality into all elements of the sequence (Radford, 2011). In relation to patterns and generalization, Mason, Burton, and Stacey (2010) highlight how students need to be able to discern an underlying general structure to be able to express a generality algebraically. Mulligan and Mitchelmore (2009), in turn, address pupils' ability to structure based on the pattern's vertical, horizontal and spatial structures.

Radford (2006) and Venenciano and Dougherty (2014) highlight the different strategies students use when making pattern generalization. Radford separated different generalization strategies in relation to how advanced they are. First of all, he made a distinction between the so-called "naive induction" versus generalizations. In the "naive induction" strategy, the students use a "trial and error strategy," which can be described as a guessing strategy and thus, according to Radford (2006), is not a generalization strategy at all. A generalization strategy is about discerning and using a general commonality of a pattern (Radford, 2006). The strategies that are counted as generalization strategies consist of both arithmetic and algebraic ones. The difference between those, according to Radford (2006), is that an arithmetic strategy does not make it possible to predict any term in a pattern as an algebraic strategy could. In other words, a generalization like "It constantly increases with two matches" is seen as an arithmetic generalization, since it only supports the prediction of the "next" positions in the sequence and does not make it possible to predict any term in the pattern.

The algebraic strategy is divided into three different strategies: factual, contextual, and symbolic. They are all categorized as algebraic since the students using these strategies are expressing a commonality that can be applied to all terms in the pattern, and thus used to predict the number of elements in any term in the pattern (Radford, 2006). Here there is not only an increase in the number of elements between the terms perceived, but the number of elements of each term is, rather, related to the position of the term in a pattern sequence (such as "the n:th term") and to all elements in the visible pattern (Radford, 2006).

The difference between those strategies is about *how* the generalization is expressed. In a factual generalization, the indeterminacy remains unnamed, and the "generality rests on *actions* performed on numbers; *actions* are made up of words, gestures and perceptual activity" (Radford, 2006, p.16). The generalization is here based on actions in relation to facts on *a* local term, 'If it's *term 1*, I *did one row*', and is then put in relation to the other terms in the sequence '*term 2*, it's *two'*, *term 3*, it´s *three*'.

Contextual and symbolic generalizations address a more mathematical level of generalization. In the contextual generalization, on the contrary to a factual generalization, the indeterminate is "made linguistically explicit: it is *named"* (Radford, 2006, p. 16). The generalization is, in other words, symbolized by words 'you *double* the *terms number'*. The difference between a contextual generalization and a symbolic generalization is that a symbolic one is based on algebraic symbols, such as '$2 \cdot x$' instead of words 'you *double* the *terms number'*.

Venenciano and Dougherty (2014) highlight another kind of strategy as algebraic. It is a measuring strategy where, for example, two squares are used as a measurement unit (see figure 2) and this puts the number of measurement units in relation to where the term is positioned in the pattern sequence.

> From a measurement approach […] one may view the unit of measure as a composite of the two squares, that which is iterated with each successive figure. This […] approach enables one to apply the notion of defining a unit and consider a scale factor to solve the problem.
>
> (Venenciano & Dougherty, 2014, p. 23)

It is argued that the teaching of algebra should give the students the opportunity to use algebra as a tool for characterizing and understanding mathematical structures (e.g., Greer, 2008; Usiskin, 1988). Additionally, a focus on making generalizations in relation to mathematical patterns is advocated in the early grades (e.g., Radford, 2006, 2011). Distinctions in relation to different types of algebraic generalisations (e.g., Radford, 2006), opens up for a broader understanding in relation to generalizations. What is lacking is descriptions of aspects which students simultaneously need to discern and take into consideration in order to be able to express and justify pattern generalization algebraically. Hence, the main object of this paper is neither about what an algebraic generalization is, nor which strategies students may use. The aim of this paper is to describe the emergence of the ability to express and justify pattern generalization algebraically. The research questions for this paper are: "What are students' qualitatively different ways of seeing pattern generalization?" and "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?"

3. Theoretical framework

Variation theory (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) has been used as a theoretical framework in this study. Learning in a Variation theoretical perspective is considered to arise in the relationship between the one who is learning and what is to be learned (Marton & Booth, 1997, see also Marton, 2015). Variation theory provides theoretical tools for the analysis of the conditions of qualitatively different ways of seeing specific knowledge, and what aspects that are critical to discern in order to be able to see this knowledge in a more powerful way. Variation in relation to a Variation theoretical perspective refers to a meaningful, conscious, directed and systematic variation of content. Critical aspects are aspects that the students need to discern to be able to develop this specific knowledge (Marton, 2015). In this paper we are exploring students' quality different ways of seeing pattern generalization. In a Variation theoretical perspective 'ways of seeing' are seen in relation to what aspects the students are discerning and focusing upon in relation to a demarcated knowledge (Marton, 2015).

4. Methodology considerations

This study is included in a more extensive practice-based research project, in which Learning study (Marton & Booth, 1997; Marton, Runesson & Tsui, 2004) is used as a research approach. This paper does not present the final results of the Learning study, but rather the analysis of semi-structured interviews, conducted initially in the study.

The semi-structured interviews

One of the first steps in a Learning study is the mapping of the students' current perceptions of a specific knowledge. In this research project semi-structured interviews were chosen as a mapping tool to grasp the students' qualitatively different ways of seeing pattern generalization and to identify aspects that students need to discern to be able to express and justify pattern generalization algebraically.
The semi-structured interviews were performed with eight of the students from the overall project. The students were 9-10 years old, and both girls and boys were interviewed. The idea was that this selection of students would cover much of the diversity that existed within the group (Marton & Booth, 1997). The students were selected in relation to their previous results in mathematics and were supposed to represent students with different performances in the subject of mathematics. The selected students were divided into pairs and were then, in the interview situation, presented with three different pattern tasks which they were asked to solve together. While the students were working with the tasks, the interviewer asked question such as "Can you tell me how you´re thinking?" and "Can you show me how you are looking at it (pointing at the pattern) when you´re saying this?". The aim was trying to explore the students' ways of seeing pattern generalization in the process of solving tasks where making pattern generalization were required. The idea was not about how pattern generalization may be defined by the students and thus was the interviewer not supposed to ask any direct questions about pattern generalization per se.

Analysis

The data in this paper consists of transcriptions of the interviews. In the analysis, Variation theoretical tools were used (critical aspects and variation of content), in order to try to distinguish qualitative dimensions of the variations in different ways of seeing pattern generalization and in relation to identifying critical aspects of the ability to express and justify pattern generalization algebraically. In the analysis, there was an interplay between the data and previous research (e.g., Radford 2010). The process of the analysis was as follows:

1. *Reading of compiled interviews*. The transcribed interviews were compiled in a running document without markings for which student said what. The document was then read several times without making any markings on the document. The aim was to try to understand what different students were saying in relation to what other students were saying.
2. *Analysis of what the students talked about.* The transcripts were read again, this time with the intention of marking those excerpts where the students talked about pattern generalization. The excerpts of the transcriptions where the students did not talk about pattern generalization were identified and removed.
3. *Analysis of how the students were talking about pattern generalization.* The excerpts where the students talked about pattern generalization were repeatedly read through a so-called comparative reading (Marton, 1995). The aim was to distinguish between the dimensions of variations of students' ways of seeing pattern generalization that were realized through students' expressions.
4. *Categorization of the students' ways of seeing pattern generalization.* Different excerpts of the students' expressions were marked with the aim to identifying qualitatively different ways of seeing pattern generalization. Those excerpts were analysed, in relation to what the students emphasized and what they seemed to discern and focus upon in relation to pattern generalization.
5. *Identifying critical aspects regarding the ability to express and justify a pattern generalization algebraically.* In the identifying process the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does this generalization work to predict any figure in the pattern?" (Radford, 2006)

5. Findings

The findings consist of two parts. Part one answers the research question "What are the students' qualitatively different ways of seeing pattern generalization?". It consists of four categories. This result is in relation to point 1-4 in the analysis.
Part two answers the research question "What aspects do students need to discern to be able to express and justify pattern generalization algebraically?". It consists of identified critical aspects regarding the ability to express and justify a pattern generalization algebraically. This result is in relation to point 5 in the analysis.

Part one - Students' qualitatively different ways of seeing pattern generalization

In the following, there are descriptions of the categories which contain student's expressions. Each category is summarized in relation to which aspects of pattern generalization that the students seemed to discern and focus upon.

...as some kind of grouping structures
In this category, the students emphasize the grouping of quantities in the sense of using a structure as a strategy to see how the pattern is built. Students, for example, undertake groupings based on the number of elements in a term (see figure 1): "... Term 1 has three (matches) and (pointing to term 2) has six (matches) ...".

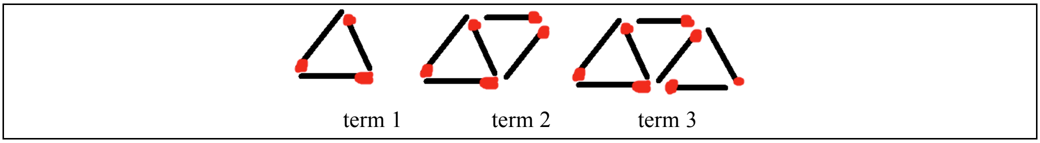

Figure 1. Matches

The students additionally grouped by adding together the number of elements in the visible terms in the pattern (Pattern 1): "... if all the matches up to term 3 is fifteen, then if you take this three, plus this three (the student is talking about the terms 1-3), it is term 6, then it is thirty matches, fifteen plus fifteen is thirty (here the student is talking about the number of matches in terms 1-3)". A characteristic of this category is that the grouping is used rather as a statement, not to predict the number of elements in a specific term.

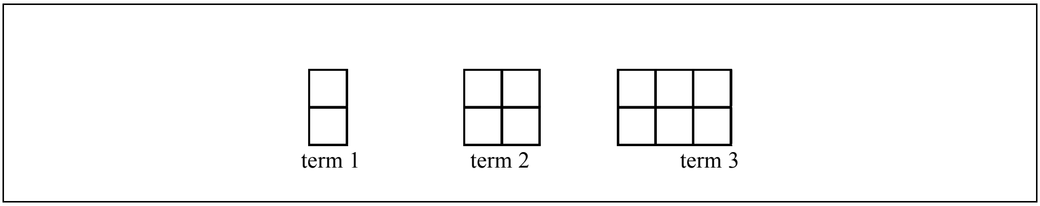

Figure 2. Squares

The students in this category seemed to discern and focus upon the following aspect: that there is a structure to follow that involves grouping objects.

...as additive constant structures
In this category, the students emphasize the adjacent terms in the sequence and the number of elements or units by which the pattern is growing. Students calculate the difference between two terms in a given sequence and distinguish this difference as being the same between all the terms. They conclude that the growth of the pattern is according to an additive structure. Based on the pattern in figure 2, a student expresses how to create the next term in the sequence: "... always add two (squares)." Other students use the column of two squares in the pattern as an integral unit, which they use as a rate of growth of the pattern: "... you only add one of those (pointing to a column of two squares)." The students see the growth of the pattern as "jumps" in the addition table: "... here are two, here are four, six, and the next eight and then it's ten."

The students in this category seemed to discern and focus upon the following aspect: the additive structure of the pattern and what constitutes the so called expansion unit (mathematical) of the pattern.

...as one dimensional relational structures

In this category the students emphasize one dimension of the pattern generalization; the number of elements or units in relation to the position of the term. The following student uses a column of two squares as a unit in relation to figure 2: "When it is term 1, I make one line (shows as a column) when it is term 2 it is two, if it is term 3 it is three columns". Another student expresses the connection between the term and the number of units (pattern 2): "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns ".

The students in this category seemed to discern and focus upon the following aspect: the relationship between the position of the term and its units.

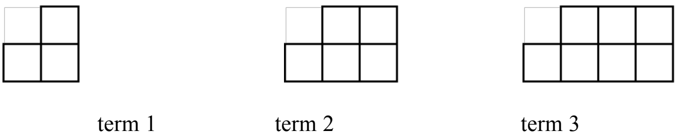

Figure 3. Squares in other way

...as two dimensional relational structures
In this category the students emphasize two dimensions of the pattern generalization; the relationship between the position of the term and the number of its elements or units and use it to predict a non-visible term in the pattern sequence. The following student expresses it, in relation to figure 2, like this: "Look, number 1 it is two, number 2 then it is four, number 3 is six, as it doubles everything… then you have to double forty-eight". In one of the tasks the students are supposed to determine the number of squares in term 46 (figure 3): "... is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)... then you can add forty and forty, its eighty and three plus three is six, then its forty-six and then we take one (the one that the student suggested should be put away) then there will be forty-seven ... no, it is eighty-seven."
The students in this category seemed to discern and focus upon the following aspects: what is the relationship between the figures number and its units and use this relationship to predict any figure in the pattern and what constitutes the constant in the pattern.

Part two - Critical aspects regarding being able to express and justify pattern generalization algebraically

The critical aspects were interpreted in relation to the categories' descriptions and which aspects the students seemed to discern and focus upon. In the identifying process, the following questions were utilized as analytic tools: "Which of the aspects that the students seem to discern and focus upon in the categories, are aspects of expressing and justifying a pattern generalization algebraically?"; "Does the aspect enable the student to predict any figure in the pattern?" (Radford, 2006).
The categories, *as some kind of grouping structures* and *as additive constant structures,* do not encompass critical aspects in relation to *algebraic* pattern generalization. The aspect *that there is a structure to follow that involves grouping objects* is not an algebraic aspect in terms of making it possible to predict the number of elements in any term in the pattern. The structure in this case concerns 'only' the grouping of quantities. The aspects *the additive structure of the pattern* and *what constitutes the expansion unit (mathematical) of the pattern* are of general character. However, this kind of generalization only works to predict the adjacent terms, since one cannot say the number of squares of any term in the pattern, such as the thousandth.
It is primarily the categories *as one dimensional relational structure* and *as two or more dimensional relational structures* that we see as encompassing aspects of algebraic character in terms of making it possible to predict the number of elements in any term in the pattern. The aspects we identified as critical aspects in relation to be able to *express and justify pattern generalization algebraically* are:

- to discern the relationship between the term's position and its units
- to discern the relationship between the term's position and its units and to use this relationship to predict any term in the pattern
- to discern what constitutes the constant in the pattern.

Here is a short description of why we consider these aspects to be critical. By discerning *the relationship between the figures number and its units* it is possible for the students to use this relationship to predict the number of squares of any term in the pattern. "... if it is 4 (term 4) it is also four columns and if it is 5 (term 5), it is also five columns." In other words, the discerned relationship is the same throughout the whole pattern, and it doesn´t matter if you are talking about term 5 or if you are talking about term 1 000. When the students discern *the relationship between the term's position* and *its units and use this relationship to predict any term in the pattern*, the students both discern the relationship "number 1 it is two" and transform this relationship "as it doubles everything" so it can be used to predict any term in the pattern. Regarding the aspect what *constitutes the constant in the pattern*, the students discern that this unit, the constant, is the same through the whole pattern "is it ok to say that if you put away this one (the constant, i.e., the lonely square to the left in each term in the pattern)?". In other words, the constant is containing the same number of units in any term if the constant is one unit in term 1 it is also one unit in term 1 000.

6. Concluding discussion

The aim of this paper was to describe the emergence of the ability to express and justify pattern generalization algebraically. In the following we will put the categorization and identified aspects of this paper in relation to other research, and mainly Radford's distinction between arithmetic and algebraic generalization strategies. The difference is, according to Radford (2006), that an arithmetic strategy does not make it possible to predict any term in a pattern, which would otherwise be the case with an algebraic strategy. The main contribution of this paper, in relation to Radford's categories, lies in the specification of critical aspects regarding what students need to discern in their learning of how to express and justify pattern generalization algebraically.

In the category *pattern generalization as additive constant structures* the students seem to discern *the additive structure of the pattern* and/or *what constitutes the expansion unit (i.e., mathematical unit) of the pattern*. Students calculate the difference between two terms in a given sequence and identify this difference as being the same between all the terms, concluding that the growth of the pattern is according to an additive structure. There might be a qualitative difference between the expressions "... always add two (squares)," where the students talk about the growth of the pattern, and "... you only add one of those (pointing to a column of two squares)," where the students use the columns of two squares in the pattern as an integral unit. This latter can be put in relation to the position of the term, which can be used to predict any term in the pattern (Moss & London McNab, 2011; Radford, 2006), since it can be seen as the beginning of using an expansion unit as a measurement unit (Venenciano & Dougherty, 2014). However, if the generalization stops at discerning only *the additive structure of the pattern* and/or *what constitutes the so called expansion unit (mathematical) of the pattern,* this is not enough to express and justify pattern generalization algebraically.

In relation to students in the early grades, we want to highlight how factual generalization and contextual strategies (Radford, 2006) can be seen as a starting point regarding developing an understanding of the meaning of algebraic notations. In other words, we consider Radford´s factual strategies and contextual strategies as indicating the emergence of being able to make symbolic generalizations. The difference between those strategies is about *how* the generalization is expressed. In the category *as one dimensional relational structures* a student expresses the following: "When it is *term 1*, I make *one line* (shows as a column) when it is *term 2* it is *two*, if it is *term 3* it is *three columns.*" In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a factual strategy, although the indeterminacy is unnamed, and the generalization here is symbolized by actions. We would equate this, in relation to our findings on critical aspects, as the student is discerning *the relationship between the term's position and its units.*

Contextual generalizations address a more mathematical level of generalization, the indeterminate is "made linguistically explicit: it is *named"* (Radford, 2006, p. 16). In the expression "Look, number 1 it is two, number 2 then it is four, number 3 is six, as *it doubles everything*… then *you have to double* forty-eight", the generalization is symbolized by words '*as it doubles everything'*. In relation to Radford's description of different algebraic generalization strategies, this can be seen in relation to a contextual generalization, although the indeterminacy is named, and the generalization here is symbolized by words. We would equate this, in relation to the findings on critical aspects, as the student is discerning *the relationship between the term's position and its units and to use this relationship to predict any term in the pattern*. Finally, our point is that this kind of generalization can later be transformed into a symbolic generalization, while drawing on the students' more informal way of describing it.

Acknowledgements
The work on this text has been done with support from the Department of Mathematics and Science Education, Stockholm University, and the City of Stockholm. We would like to thank the working group for constructive comments. We also want to thank Gail FitzSimons and Hauke Straehler-Pohl for their readings of earlier versions of this text.

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II

Paper 2: Att designa för elevers deltagande i ett algebraiskt arbete: Elever i årskurs 2 och 3 utforskar visuellt växande mönster

Fred, J. (2019a). Att designa för elevers deltagande i ett algebraiskt arbete: Elever i årskurs 2 och 3 utforskar visuellt växande mönster. *Nordisk matematikkdidaktikk NOMAD[Nordic Studies in Mathematics Education]*, *24*(3-4), 107–130.

This article discusses and analyzes different lesson sequences, with a focus on creating conditions for students in grades 2 and 3 to engage in algebraic work and thus provides opportunities to identify several critical aspects of the ability to express and argue for algebraic pattern generalizations. The following research question was addressed: In what ways can didactic principles for learning activities support the establishment and maintenance of algebraic work and thus enable a discernment of critical aspects?

The article presents the following results, focusing on the didactic principles for learning activities. The *problem situation* did not state that the students should compare the structural properties of the given elements; rather, it invited them to do so. The problem situation seemed to create conditions for the students to discern how they could use the relationship between the position of an element and the number of components to predict an arbitrary element in the pattern. This was done because the problem situation invited a comparison of the structural properties of elements.

The results showed how a *learning model* served as a didactic tool in maintaining algebraic work. This happened when the students collectively engaged in a further exploration of the algebraic structures of a visually growing pattern. In the sequence, the learning model had previously taken the form of encirclements of components in given elements (see Figure 2). This happened when the students were given the task of marking, for each given element, how they had used it to determine a non-given element.

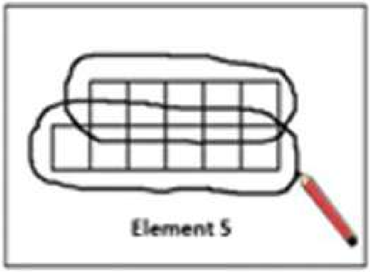

Figure 2 A learning model

The two markings in element 5 of Figure 2—one marking of the row above and the other from the row below—together constitute a learning model. This learning model gave other students access to the critical aspect—the relationship between the element's position and the number of components—by allowing the components to be grouped around the element's position in the pattern. Furthermore, the learning model was a tool for classroom communication and reflective discussions. This happened, for example, when the teacher incorrectly converted the component grouping into numbers, prompting students to argue about what was correct or incorrect in the transformation. When the teacher later offered an alternative structure by changing the circles in each element, students discerned that the relationship between an element's position and the number of components can be expressed in different ways. Thus, the results also showed how the teacher's and students' interactions about creating and establishing the learning model, and about the contradictions encountered, enabled the establishment and maintenance of algebraic work.

The results showed examples of how the didactic tool of *contradictions* was important for maintaining algebraic work. This became apparent when students were invited to support their arguments by addressing aspects such as the relationship between the element's position and the number of components. Furthermore, the article gave examples of how a planned contradiction allowed the students to develop a more general understanding of how the relationship between the positions of elements and the number of components can be formulated

Att designa för elevers deltagande i ett algebraiskt arbete

Elever i årskurs 2 och 3 utforskar visuellt växande mönster

JENNY FRED

Artikelns syfte är att beskriva och analysera vad i olika lektionssekvenser som skapar förutsättningar för att elever ska engageras i ett algebraiskt arbete och därmed urskiljer kritiska aspekter. Artikeln bygger på data från tre forskningslektioner i vilka lärandeverksamhet (*learning activity*) tillsammans med Radfords arbete om mönstergeneraliseringar har utgjort teoretiska utgångspunkter. I analysen har didaktiska principer från lärandeverksamhet samt kritiska aspekter gällande att uttrycka och argumentera för mönstergeneraliseringar fungerat som analysredskap. Resultatet kan bidra till att fördjupa förståelsen gällande på vilka sätt principerna från lärandeverksamhet kan stödja ett etablerande och upprätthållande av ett algebraiskt arbete och därmed möjliggöra för elevers urskiljande av kritiska aspekter.

Inom forskningsfältet som idag benämns *early algebra* (se t ex Cai & Knuth, 2011; Kaput, 2007; Lins & Kaput, 2004; Radford, 2006, 2010a, 2010b, 2014) argumenteras för att elever redan i de lägre årskurserna behöver introduceras till ett algebraiskt arbete[1], för att därmed stödja framväxten av ett algebraiskt tänkande. Elever i yngre åldrar som engageras i ett algebraiskt arbete förväntas fokusera på att uppmärksamma, generalisera, kommunicera och argumentera för matematiska strukturer. Följaktligen behöver inriktningen i undervisningen riktas mot matematiska strukturer snarare än på bemästrandet av matematiska operationer (se t ex Radford, 2012).

Mönster och generaliseringar föreslås som en möjlig ingång till algebra i de lägre årskurserna (Mason, 1996; Radford, 2006; Warren, 2005). Matematiska mönster har en inneboende struktur gällande hur mönstrets olika delar är organiserade som kan vara föremål för ett utforskande arbete av strukturer och generaliseringar (Mulligan & Mitchelmore,

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2009). En undervisning som tar sin utgångspunkt i mönster med fokus på matematiska strukturer och generaliseringar antas främja framväxten av elevers algebraiska tänkande (se t ex Blanton & Kaput, 2011; Radford 2006).

Undervisning utgörs av ett komplext samspel mellan uppgifter, instruktioner, orkestrering av elevers arbete, kunskapsinnehållet samt eleverna och läraren (Lindberg, 2010; Newman, Griffin & Cole, 1989). I relation till uppgiftsdesign framhåller Strømskag (2015) betydelsen av själva frågeställningen i uppgiften och vad den riktar elevers uppmärksamhet mot. Vidare betonar Strømskag att uppgifter som endast fokuserar antalet komponenter i mönster, och inte dess inneboende multiplikativa strukturer, inte är gynnsamma för utvecklingen av algebraiskt tänkande. I relation till orkestrering av elevers arbete framställs bland annat lärares förmåga att identifiera, agera och utnyttja betydelsefulla matematiska situationer som uppstår i undervisningen (Blanton & Kaput, 2011; Larsson, 2015; Warren, 2005). Vidare betonar Larsson (2015) behovet av medierande redskap för att kunna hantera helklassdiskussioner som tar sin utgångspunkt i elevers olika idéer. Ryve, Larsson och Nilsson (2011) framhåller att elever behöver erbjudas medierande redskap för tänkande och kommunikation i form av tabeller, siffror och muntliga uttryck.

I så väl planeringen som genomförandet av undervisningen är bland annat lärares förmåga att bygga in förutsättningar för algebraisk tänkande i instruktionerna och frågorna central. Instruktioner, frågor och medierande redskap har alltså i denna artikel en central roll gällande att skapa förutsättningar för att elever engageras i ett algebraiskt arbete med visuellt växande mönsters matematiska strukturer.

I relation till det kunnande eleverna ska utveckla understryker Runesson (2011) och Kullberg (2011) betydelsen av att elever får möjlighet att urskilja det de beskriver som *kritiska aspekter*, det vill säga aspekter av ett specifikt och avgränsat kunskapsinnehåll som i relation till en specifik elevgrupp kan antas vara avgörande för deras lärande. Aspekterna anses vara kritiska då eleverna ännu inte har urskilt dem. För att avsett lärande ska äga rum behöver eleverna urskilja de kritiska aspekterna. Att i design av undervisning ta utgångspunkt i kritiska aspekter gällande exempelvis ett avgränsat algebraiskt kunskapsinnehåll har i tidigare studier visat sig vara gynnsamt för elevers lärande (se t ex Fred & Stjernlöf, 2015).

För att designa en undervisning som engagerar elever i ett algebraiskt arbete krävs kunskaper om uppgiftsdesign och design av orkestrering av klassrumsarbetet som kan skapa förutsättningar för att elever engageras i ett algebraiskt arbete. Vidare krävs kunskaper gällande det kunskapsinnehåll som undervisningen ska riktas mot. Syftet med denna artikel är att beskriva och analysera vad i olika lektionssekvenser som skapar förutsättningar för att elever ska engagera sig i ett algebraiskt arbete och därmed få möjlighet att urskilja ett antal identifierade kritiska aspekter gällande förmågan att uttrycka och argumentera för algebraiska mönstergeneraliseringar (Fred & Boistrup, 2017, se vidare nedan). Följande fråga adresseras: På vilka sätt kan didaktiska principer för lärandeverksamhet stödja ett etablerande och ett upprätthållande av ett algebraiskt arbete och därmed möjliggöra för ett urskiljande av kritiska aspekter?

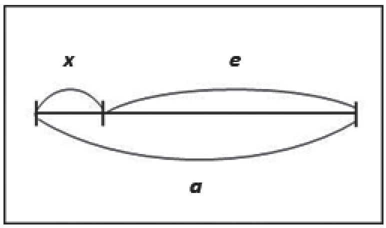
Teoretiska utgångspunkter

Artikeln bygger på data som utgår från Davydovs (2008) lärandeverksamhet (*learning activity*) tillsammans med Radfords arbete (t ex 2006, 2010a, 2010b, 2011) om mönster och mönstergeneraliseringar som teoretiska utgångspunkter.

Lärandeverksamhet

Lärandeverksamhet (Davydov, 2008) är ett teoretiskt och didaktiskt ramverk som har sin grund i Vygotskijs (2001) kulturhistoriska perspektiv och Leontievs (1978) verksamhetsteori. I ett kulturhistoriskt perspektiv ses lärande som ett förändrat deltagande (handlande) i en specifik praktik som är kulturellt och historiskt utvecklad (Davydov, 2008). Bemästrande av praktikspecifika redskap ses som en aspekt av vad det innebär att vara kunnig. Vidare betonas att elever för att lära sig behöver försättas i problemsituationer som utmanar till ett teoretiskt arbete, i vilket eleverna själva är aktiva agenter (Eriksson, 2017). Intentionen med en lärandeverksamhet är att skapa förutsättningar för att elever ska kunna engagera sig i ett teoretiskt arbete där de kollektivt och aktivt konstruerar och utvecklar systematiska och vetenskapliga begrepp inom specifikt kunskapsinnehåll (Davydov, 2008; Repkin, 2003). Lärandeverksamhet tillhandahåller principer om hur undervisning, uppgifter och klassrumskommunikation kan designas så att elever engageras i ett teoretiskt (läs här algebraiskt) arbete. Tre centrala didaktiska principer är 1) skapandet av innehållsliga *problemsituationer*, 2) skapandet och etablerandet av *lärandemodeller* och 3) skapandet eller framlyftandet av *motsättningar* (Eriksson, 2017, se även Björk, Nikula, Stensland & Stridfält, 2019; Eriksson m fl, 2019)

I lärandeverksamhet har lärandemodeller en central betydelse gällande att möjliggöra för elever att kollektivt engagera sig i ett teoretiskt arbete. Det är i målet att lösa ett givet problem som arbetet med lärandemodellen tar form. Idén är att lärandemodellen ska fungera som ett redskap i det kollektiva teoretiska arbetet genom att lärandemodellen skapas, rekonstrueras och/eller bearbetas (Davydov, 2008; Gorbov & Chudinova, 2000). Lärandemodeller kan vara materiella, ikoniska eller semiotiska och tar bland annat form som scheman, skisser, sträckor eller symboler (Davydov, 2008). Centralt är att lärandemodellen fångar teoretiska aspekter av det kunskapsinnehåll som ska läras och att det sedan är genom att eleverna skapar, rekonstruerar och bearbetar lärandemodellen som de får tillgång till kunskapsinnehållet (Eriksson, 2018; Gorbov & Chudinova, 2000). I figur 1 visas ett exempel på en lärandemodell.



Figurl. *Exempel på lärandemodell* [2]

Lärandemodellen i figur 1 syftar till att gestalta relationen mellan *a*, *x* och *e* och intentionen är att den därmed ska möjliggöra för eleverna att genom ett teoretiskt arbete utforska strukturer och relationer i algebraiska uttryck.

Motsättningar är historiskt utvecklade spänningar som i vardagen ofta kommer till uttryck i form av upplevda konflikter, dilemman eller en hake (Eriksson, 2017). Vidare är idén med motsättningar att de kan användas som redskap i designen av problemsituationer för att på så sätt utmana elever att engageras i ett teoretiskt arbete (Davydov, 2008; Zuckerman, 2003). Det är när motsättningen framträder i form av en hake eller liknande som den kan fungera som en drivkraft i elevers teoretiska arbete. Motsättningar beskrivs även utgöra grunden för att en praktik förändras eller utvecklas, då eleverna i sina försök att överbrygga motsättningar behöver hitta nya redskap och metoder (Eriksson, 2017). Vidare är intentionen att försöka planera motsättningar för att stimulera elever till ett ytterligare utforskande av innehållsliga aspekter. Till exempel kan motsättningar handla om att läraren eller någon elev skapar ett matematiskt dilemma eller en hake som behöver hanteras och som när det hanteras möjliggör för ett urskiljande av innehållsliga aspekter.

Även idén om kollektiva reflektioner är centralt i lärandeverksamhet, där reflektion ses som en viktig aspekt i utvecklingen av elevers framväxande teoretiska tänkande. Med kollektiv reflektion avses i lärandeverksamhet att man försöker förklara hur någon annan har tänkt och resonerat. Idén är således att eleverna ska utvecklas i sitt eget teoretiska tänkande genom att försöka reflektera över hur någon annan har tänkt och genom att sätta det i relation till hur man själv tänker (Zuckerman, 2004).

Algebraiska mönstergeneraliseringar

Radfords (2006) distinktion mellan aritmetiska och algebraiska mönstergeneraliseringar handlar inte om huruvida skriftliga symboler används eller ej, utan på vilket sätt det generella i ett mönster behandlas. I stora drag handlar skillnaden om att en aritmetisk mönstergeneralisering inte gör det möjligt att förutsäga vilket element som helst i ett mönster, något som däremot en algebraisk mönstergeneralisering gör. Exempelvis ses en mönstergeneralisering som ”nästa element har två fler” som aritmetisk, då den representerar endast den konstanta differensen mellan två element. En aritmetisk mönstergeneralisering anses alltså inte ge underlag för att förutsäga relationen mellan ett godtyckligt elements position och antalet komponenter. En mönstergeneralisering som ”addera tre till elementets nummer” anses däremot algebraiskt, då den uttrycker relationen mellan elementets position och antalet komponenter (Fred & Boistrup, 2017; Radford, 2006).

Radford (2006) delar in algebraiska mönstergeneraliseringar i tre kategorier: verklig, kontextuell och symbolisk baserat på sättet de uttrycks på. En verklig mönstergeneralisering baseras på handlingar i form av ord, gester eller perceptuell aktivitet. Den kan ta form genom att eleven urskiljer specifika elements interna relationer, att till exempel eleven pekar på respektive element samtidigt som han/hon uttrycker dess uppbyggnad verbalt ”Det skulle vara ett plus tre /... / två plus tre /... / tre plus tre” (min översättning, Radford, 2006, s 10). I en kontextuell mönstergeneralisering har det obestämda gjorts språkligt explicit, det namnges. Det innebär att mönstergeneraliseringen kommer till uttryck i verbala utsagor, som till exempel ” Du dubblar siffran på elementet” (min översättning, Radford, 2006, s 13). I en symbolisk mönstergeneralisering baseras mönstergeneraliseringen på det matematiska symbolspråket, till exempel ”$n + n$” (Radford, 2006, s 13).

Metod och data

Learning study (Marton & Tsui, 2004; Marton, 2015) har använts som ansats för dataproduktion.[3] I denna learning study arbetade artikelns författare och lärare tillsammans i en kollaborativ och intervenerande process med att utveckla en lektion. Intentionen var att förfina undervisningsdesignen för en och samma lektion och genom det studera vad som skapade förutsättningar för elever att engageras i ett algebraiskt arbete och därmed ges möjlighet att urskilja ett antal identifierade kritiska aspekter gällande förmågan att uttrycka och argumentera för algebraiska mönstergeneraliseringar. I arbetet med att utveckla lektionen genomfördes tre forskningslektioner där respektive forskningslektion genomfördes i en ny elevgrupp. I samtliga tre forskningslektioner var artikelns författare den undervisande läraren.[4] Denna learning study genomfördes i en iterativ process i tre iterationer där varje iteration innehöll stegen: design, genomförande samt analys och revidering av forskningslektion.[5] Analysen av forskningslektionerna genomfördes i två steg (jfr Carlgren, Eriksson & Runesson, 2017), där steg 1 skedde mellan forskningslektionerna och steg 2 efter att samtliga tre iterationer genomförts. I denna artikel presenteras framför allt resultat från den fördjupade analysen i steg 2. Det datamaterial som ligger till grund för artikeln beskrivs i tabell 1.

Tabell 1. *Datamaterial*

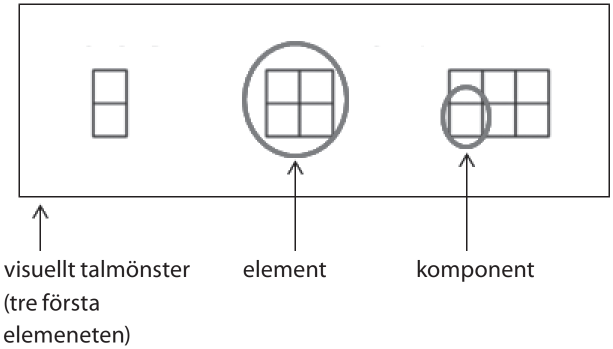
Data	Antal	Sidor	Minuter
Lektionsplaneringar	3	12	
Videoinspelade forskningslektioner	3		111
Transkriberade forskningslektioner	3	13	

Samtliga elever i årskurs 2 och 3 fick genom sina vårdnadshavare en förfrågan om att delta i forskningsprojektet. Elever fick även i samband med forskningslektionerna ge sitt muntliga medgivande. Sammanlagt deltog 53 elever i forskningslektionerna.

De elever som finns med i utdragen i denna artikel har getts fingerade namn, anonymiserats genom att de har fått ett annat namn, dock har flickor fått flicknamn och pojkar fått pojknamn.

Design av forskningslektioner

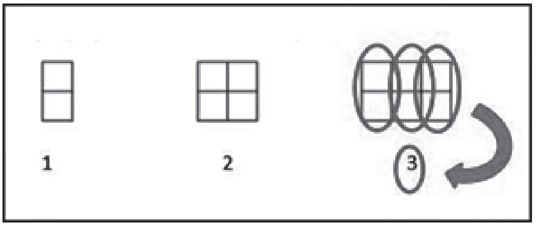
De mönster som användes i forskningslektionerna var visuellt växande mönster (se figur 2) som illustreras med bilder, där komponenterna i elementen var i form av kvadrater, och som ökar eller minskar enligt en additiv struktur (Warren & Cooper, 2008). I artikeln benämns ett mönsters olika delar enligt bilden (figur 2) och benämningen tar sin utgångspunkt i Strømskag Måsøval (2011, s 140):



Figur 2. *Benämning av ett mönsters olika delar*[6]

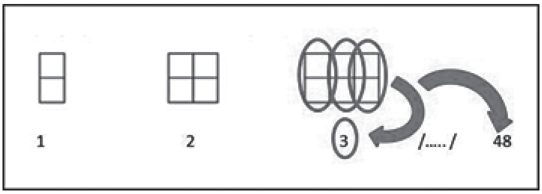
I den didaktiska designen fungerade begreppen *problemsituation*, *lärandemodell* och *motsättningar* som designredskap (Davydov, 2008).[7] Vidare fungerade ett antal identifierade kritiska aspekter gällande förmågan att uttrycka och argumentera för algebraiska mönstergeneraliseringar som utgångpunkt i designen (Fred & Boistrup, 2017). De kritiska aspekterna hade identifierats i en kartläggningsstudie[8] (Fred & Boistrup, 2017), där elevers kvalitativt skilda sätt att erfara mönstergeneraliseringar utforskades. Kartläggningsstudien genomfördes före första forskningslektionen. Den utgjordes av fyra semistrukturerade intervjuer, i vilka åtta elever parvis fick arbeta med uppgifter som efterfrågade att de gjorde mönstergeneraliseringar. Med utgångspunkt i transkriptioner av elevintervjuerna kategoriserades sedan elevers skilda sätt att uppfatta mönstergeneraliseringar, vilka i sin tur utgjorde underlag för identifieringen av följande kritiska aspekter.

A *Att urskilja relationen mellan elementets position och antal komponenter*
Eleverna betonar här antalet komponenter i förhållande till elementets position vilket indikerar att de endast urskiljer enskilda elements interna strukturella relation, det vill säga relationen mellan elementets position och antalet komponenter.



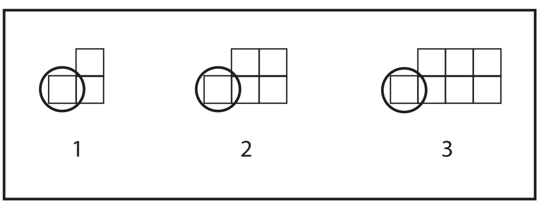
Figur 3. *Relationen mellan elementets position och antalet komponenter*

B *Att urskilja hur man kan använda relationen mellan ett elements position och antalet komponenter för att förutsäga ett godtyckligt element i mönstret*
Eleverna urskiljer här såväl elementens interna som externa strukturella relationer, det vill säga de urskiljer både relationen på enskilda elements position och antalet komponenter samt hur de enskilda komponenternas relation kan sättas i relation till varandra. Relationen används också för att förutsäga ett godtyckligt element i mönstret.



Figur 4. *Att använda relationen mellan ett elements position och antalet komponenter för att förutsäga ett godtyckligt element i mönstret*

C *Att urskilja konstanten i mönstret*
Eleverna urskiljer här den eller de komponenter som inte förändras i mönstret. Urskiljandet av konstanten kan till exempel komma till uttryck "... då kan man säga att om man lägger undan den ... [pekar samtidigt på den markerade kvadraten i mönstret i figur 5]". Med andra ord består konstanten av lika många komponenter oberoende av elementets position.



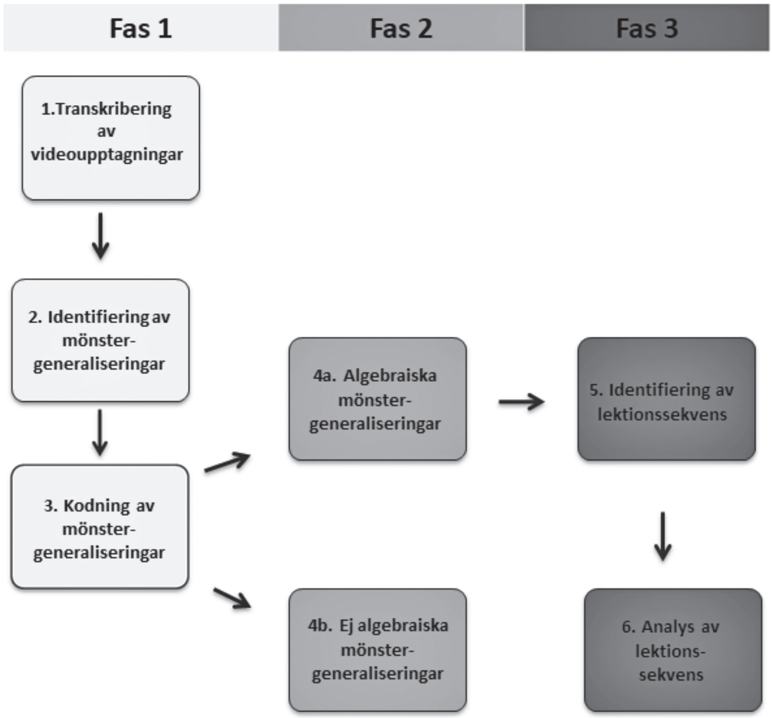
Figur 5. *Konstanten i mönstret*

Analysprocess

Analysen har skett i tre faser, där varje fas innehåller olika steg (figur 6).

Fas 1

I fas 1 ingick följande steg (1) *transkription av videoupptagningar,* (2) *identifiering av mönstergeneraliseringar* (3) *kodning av mönstergeneraliseringar.* I steg 1 transkriberades delar av videoupptagningar från forskningslektion 1 och 2 samt hela forskningslektion 3. Valet av vilka delar som transkriberades tog sin utgångspunkt i de synopsis som gjorts på samtliga



Figur 6. *Analysprocessen*

forskningslektioner. I steg 2 identifierades och markerades i transkriptionerna elevernas uttryckta mönstergeneraliseringar. I steg 3 kodades uttryckta mönstergeneraliseringar med utgångspunkt i de kritiska aspekter eleverna gav uttryck för att urskilja.

Fas 2

I fas 2 kategoriserades, med kodningen som grund, de uttryckta mönstergeneraliseringarna i *algebraiska mönstergeneraliseringar* (steg 4a i figur 6) respektive *ej algebraiska mönstergeneraliseringar* (steg 4b i figur 6). I kategorin ej algebraiska mönstergeneraliseringar kategoriserades mönstergeneraliseringar vilka ej föll inom ramen för de tidigare identifierade kritiska aspekterna eller kunde identifieras som vad Radford (2006) beskriver som algebraiska mönstergeneraliseringar. Fördelningen visas i tabell 2.

Fas 3

I fas 3 ingick *identifiering av lektionssekvenser* (steg 5 i figur 6) samt *analys av lektionssekvenser* (steg 6 i figur 6). Detta arbete tog sin utgångspunkt i

Tabell 2. *Kategorisering av ej algebraiska vs algebraiska mönstergeneraliseringar*

	Ej algebraisk mönstergeneralisering	Algebraisk mönstergeneralisering
Forskningslektion 1	2	0
Forskningslektion 2	4	2
Forskningslektion 3	2	6

de åtta identifierade algebraiska mönstergeneraliseringarna (från forskningslektion 2 och 3) i fas 2. Fokus i steg 5 var att fånga lektionssekvenser som helhet. Stöd i identifieringsprocessen var frågorna: ”Vilka lektionssekvenser/uppgifter möjliggjorde ett urskiljande av kritiska aspekter?” samt ”I vilka lektionssekvenser möjliggjorde lärarens orkestrering av klassrumsarbetet ett urskiljande av kritiska aspekter?” som analysredskap. I steg 6 analyserades de identifierade lektionssekvenserna med fokus på: ”Vad i uppgiftsdesignen och lärarens orkestrering av klassrumsarbetet möjliggjorde ett algebraiskt arbete gällande utforskandet av visuellt växande mönsters algebraiska strukturer?” I analysarbetet användes innebörder av de tre didaktiska principerna från lärandeverksamhet som analysredskap: *problemsituation*, *lärandemodeller* och *motsättningar*.

I föreliggande artikel ligger fokus på att beskriva och analysera vad som skapar förutsättningar för att elever engageras i ett algebraiskt arbete i undervisningen. Det finns samtidigt i varje forskningslektion saker som har begränsat och hindrat elevernas deltagande i ett algebraiskt arbete. Då det inte har varit fokus på i artikeln har detta uteslutits från analysen.

Resultat

I resultatet nedan presenteras exempel på lektionssekvenser från forskningslektion 3 där det framträder extra tydligt att eleverna engagerades i ett algebraiskt arbete och därmed kan antas ha möjlighet att urskilja de kritiska aspekterna. Resultatet är organiserat med utgångspunkt i tre lektionssekvenser[9], innefattande (1) en beskrivning av lektionssekvensen som helhet samt (2) en analys av lektionssekvensen. Urvalet av lektionssekvenser, tre av totalt åtta identifierade, motiveras med syftet att synliggöra vilka redskap som kan ha betydelse i uppgiftens design och lärarens orkestrering av klassrumsarbetet för möjliggörandet av elevernas urskiljande av de kritiska aspekterna.

Under forskningslektionernas genomförande använde läraren ordet figur när hon talade om olika element (se figur 2) i mönster. I nedanstående utdrag har ordet figur ersatts med ordet element, detta för att ordet figur som gäller för datautdrag (bilder på uppgifter, mönster etc.) inte ska sammanblandas med benämningen figur som gäller för element i mönstren.

Lektionssekvens 1

Den första lektionssekvensen exemplifierar vad som bidrog till att eleverna urskilde den kritiska aspekten *relationen mellan elementets position och antalet komponenter.* I analysen av denna lektionssekvens adresseras framför allt samspelet mellan *uppgiftens design* och *lärarens orkestrering av klassrumsarbetet* som möjliggjorde detta urskiljande. Vidare diskuteras i analysen av denna lektionssekvens vilken funktion lärandemodellen fick.

Lektionssekvensen inleddes med att eleverna fick ett mönster (figur 7) där element 3 och element 5 var givna. Läraren introducerade uppgiften genom att säga ”Er uppgift nu är att ni ska rita element 4. Ni har fått element 3 och ni har fått element 5.”

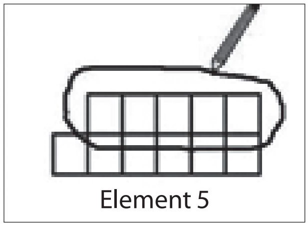
Figur 7. *Uppgift ”Rita element 4”*

Efter en kort stund avbröt läraren elevernas parvisa arbete.

Läraren: Vet ni nu är jag inte så *jätte intresserad egentligen* ... Vet ni vad jag är intresserad av? Det är *hur* ni *tittade* på de här elementen [pekar på element 3 och 5 som finns på tavlan, se figur 7] när ni kom fram till hur element 4 skulle se ut?

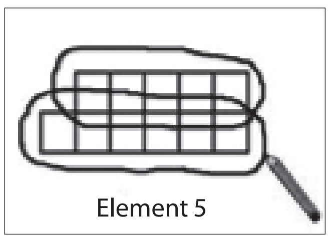
Läraren ändrade alltså nästan genast inriktningen på frågeställningen, från hur ett specifikt element skulle ritas till hur eleverna hade använt sig av de två givna elementen för att bestämma hur det icke givna elementet skulle ritas.

Eleverna fick därefter fundera ytterligare en kort stund innan läraren fortsatte med att säga: ”För jag tror att ni kan ha tittat lite olika på mönstret. Så skulle någon vilja komma fram och *rita hur* de tittade? Vill du, Thea, rita?” Thea och hennes kompis Elvira gick fram till tavlan men de såg lite tveksamma ut. Läraren visade då, genom att i luften ringa in den ensamma kvadraten längst till vänster i element 5 (figur 7), hur man kunde rita. Kort därefter började Thea ringa in den översta raden i element 5 i det mönster som fanns på tavlan (se figur 8).



Figur 8. *Eleven ringar in övre raden i element 5*[10]

När Thea hade ritat klart i element 5 tittade hon frågande på läraren. Läraren uppmanade eleverna att fortsätta genom att säga: ”Ok, hur gjorde ni sedan när ni hade sett det där [pekar på den rad Thea har ringat in]? Tittade ni på något annat?” Thea fortsatte då att ringa in den undre raden på element 5.



Figur 9. *Eleven ringar in undre raden i element 5*

Samtidigt som Thea ringade in i element 5 gick Elvira fram och pekade på först den övre raden och sedan på den undre raden i element 3. Thea fortsatte därefter med att ringa in på motsvarande sätt i element 3.

Analys

Uppgiftens design i lektionssekvensen ovan bestod av två steg. I det första steget skulle eleverna rita ett icke givet element, och i det andra steget skulle eleverna berätta hur de hade tittat på två givna element för att bestämma det icke givna elementet. Läraren skapade här en problemsituation vilken möjliggjorde för att eleverna började jämföra elementets strukturella egenskaper i mönstret. Läraren uttryckte inte explicit att eleverna skulle göra detta, men framför allt genom uppgiftens andra steg riktades elevernas uppmärksamhet mot *hur givna element kunde användas för att förutsäga ett godtyckligt element.* Att det också var det som skedde framträdde genom elevernas inringningar i elementen, då de grupperade komponenterna på ett sätt som kunde sättas i relation till elementets position.

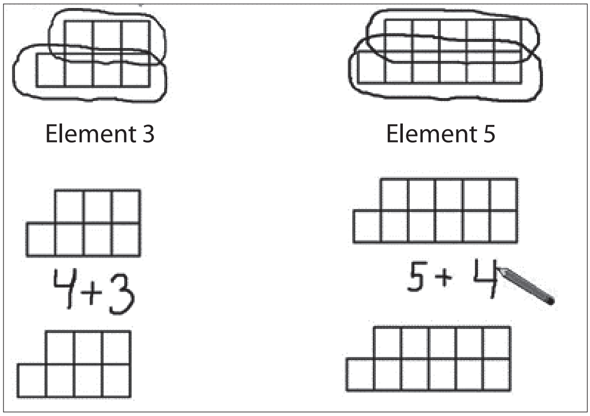
Lärarens orkestrering av klassrumsarbetet skapade även förutsättningar för att en lärandemodell tog form. Det var elevernas inringningar i givna element som fick funktionen av en lärandemodell. Den övre raden bestod av samma antal komponenter som elementets position i mönstret angav och den undre raden bestod av en extra komponent. Med andra ord kunde grupperingarna av komponenterna sättas i relation till elementets position. Eleverna fick således tillgång till, genom inringningarna, en algebraisk struktur gällande *relationen mellan elementets position och antalet komponenter.*

Lektionssekvens 2

Den andra lektionssekvensen exemplifierar hur eleverna urskilde hur *relationen mellan elements position och antalet komponenter kan användas för att förutsäga ett godtyckligt element i mönstret.* I relation till denna lektionssekvens adresseras framför allt hur eleverna, genom att läraren gjorde fel, inbjöds till att reflektera över *relationen mellan elementets position och antalet komponenter samt hur den kan användas för att förutsäga ett godtyckligt element i mönstret*.

Lektionssekvensen inleddes med att läraren av misstag omvandlade Theas och Elviras inringningar i element 5, framme på tavlan, felaktigt: ”Så som de [Thea och Elvira] här hade sett på det [mönstret]. Skulle jag kunna skriva det så här?” Läraren pekade på elementens övre respektive undre rad samtidigt som hon sa och skrev omvandlingen av inringningarna, det vill säga sa hon fem samtidigt som hon pekade på övre raden i element 5 och så vidare (se figur 10).

Den felaktiga omvandlingen innebar att inringningarna i element 5 omvandlades till ”5 + 4” istället för ”5 + 6”. Element 3 omvandlades däremot relevant till ”4 + 3”. Läraren fortsatte efter att ha omvandlat även element 3 med att ställa frågan: ”Var det så ni såg det tjejer? Eller är det något ni vill ändra?”. Thea och Elvira svarade genom att skaka på huvudena och läraren inbjöd då eleverna att uttrycka vad det var som de inte höll med om: ”Det var inte så ni såg det? [Låter förvånad i rösten – har ännu inte upptäckt sitt misstag]”. Varpå Elvira började argumentera för varför hon inte tyckte att omvandlingen var korrekt: ”Det var tre där uppe. Det är tre. På element 3 är det tre rutor. Men element 5 är det fem däruppe och då så måste det på element 4, fyra där uppe.”



Figur 10. *De tre uppsättningarna av det visuellt växande mönstret på tavlan*

Analys

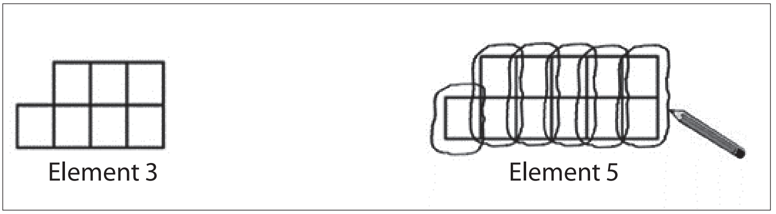
Under lektionssekvensen riktades elevernas uppmärksamhet mot de givna elementens strukturer genom att läraren uttryckte omvandlingen av Theas och Elviras tidigare grupperingar verbalt, med gester samt genom att symbolisera dem med tal.

Läraren byggde även omedvetet in en motsättning, i form av en hake, i lektionssekvensen genom att omvandla tidigare inringningar felaktigt. Motsättningen fungerade som utgångspunkt i elevernas kollektiva arbete, genom att den inbjöd eleverna att ytterligare reflektera över relationen mellan elementets position och antalet komponenter. Att det var det som också skedde blev synligt i Elviras argumentation, då hon tog stöd i *elementens position i relation till antalet komponenter* på övre raden (se ovan).

Lektionssekvens 3

Den tredje lektionssekvensen exemplifierar även den hur eleverna urskilde hur de kunde *använda relationen mellan elements position och antalet komponenter för att förutsäga ett godtyckligt element i mönstret.* I relation till denna lektionssekvens adresseras framför allt hur läraren skapade förutsättningar för detta genom att erbjuda en alternativ struktur samt att bjuda in eleverna att reflektera över den alternativa strukturen.

Lektionssekvensen inleddes med att läraren erbjöd en alternativ gruppering av komponenterna i element 5. Den alternativa grupperingen, som fanns på tavlan, innebar att läraren ringade in komponenterna lodrätt två och två och den ensamma kvadraten till vänster för sig (figur 11).



Figur 11. *Läraren erbjuder en alternativ struktur*

Läraren bjöd sedan in eleverna genom att, med utgångspunkt i den alternativa grupperingen, berätta hur element 5 skulle skrivas:

Läraren: Om jag såg mönstret så [pekar mot inringningarna i element 5]? Hur skulle jag skriva då, det här [låter undrande]? Här skrev jag fem plus sex [pekar på tidigare inringningar i element 5].

En elev, Elton, svarade snabbt: ”Två plus två plus två plus två plus två plus en.” Varpå läraren bad honom att förtydliga med hur många tvåor han sa. Elton svarade först på frågan hur många tvåor han hade sagt och gav sedan ett alternativt förslag för hur element 5 skulle kunna uttryckas: ” Fem. Eller två gånger fem plus 1.” Efter en stund flyttade läraren fokus från element 5 till hur ett icke givet element skulle kunna uttryckas: ” Om vi ska skriva hur element 7 är uppbyggd då? Hur ska vi skriva då?” Samtidigt som läraren frågade detta påminde hon eleverna om hur uppbyggnaden av element 5 uttryckts tidigare: ”När vi tittade på figur 5 då skrev vi 2 + 2 + 2 + 2 + 2 + 1 eller 5 + 6.” Ella svarade genom att uttrycka hur elementets komponenter kunde grupperas horisontellt på ett sätt som gick att sätta i relation till elementets position: ”Sju där uppe och åtta där nere.” Läraren bad därefter de andra eleverna att beskriva vad i Ellas verbalt uttryckta grupperingar som de höll med om och varför: ”Håller ni andra med om det? Varför håller ni med? Vad är det ni håller med om?” Varpå Sophie valde att argumentera med utgångspunkt i grupperingarna av de övre raderna i element 3 och element 5: ”På element 3 så är det tre där uppe och på element 5 är där fem där uppe, så på element 7 måste det vara sju.” I nästa skede kopplade läraren tillbaka till den alternativa grupperingen som uttryckte lodrätta grupperingar av komponenterna.

Läraren: Om man vill använda det här sättet då?

Elton: Då kan man ta sju tvåor plus 1.

Läraren: Håller ni med om det här då? Om det är element 7 kan man skriva så? Ska det vara sju tvåor plus ett? Eller kan man skriva 2 x 7 + 1? Och varför håller ni med? Vad är det ni håller med om?

Grete: För att på element 5 så är det 2 + 2 +2 + 2 + 2 +1 och där var det ju fem (---) och då blir det ju fem tvåor men på figur sju blir det ju sju tvåor. Då varierar sju-siffrigt (---).

Analys

I lektionssekvensen ovan skapade läraren förutsättningar för att eleverna skulle kunna urskilja att mönstrets algebraiska struktur kunde uttryckas på olika sätt, detta genom att hon skapade en motsättning mellan en lodrätt gruppering och Theas och Elviras horisontella gruppering. Eleverna behövde således hantera de båda grupperingarna samtidigt och också ytterligare utforska hur komponenterna hade grupperats i relation till elementens position. Mot bakgrund av detta blev det möjligt för eleverna att utveckla en mer generell förståelse gällande hur *relationen mellan elements position och antalet komponenter* kan formuleras. Det var genom att de olika grupperingarna båda visualiserade en relation mellan elementens position och antalet komponenter men samtidigt såg och uttrycktes olika som skapade förutsättningar för detta. Till exempel uttrycktes element 5 som ”5 + 6” eller som ”två gånger fem plus 1”.

Lärarens orkestrering av klassrumsarbetet bestod i att läraren genom att ställa frågan: ”Om vi ska skriva hur element 7 är uppbyggd då? Hur ska vi skriva då?” bjöd in eleverna att använda sig av hur elements uppbyggnad tidigare hade uttryckts. Med andra ord använde läraren den interna strukturen för element 5 som redskap för att eleverna skulle kunna uttrycka ett icke givet element. Läraren riktade samtidigt elevernas uppmärksamhet mot att det var just strukturen som skulle fokuseras, detta genom att efterfråga elementets uppbyggnad.

I denna lektionssekvens innebar även lärarens orkestrering att hon genom att ställa frågorna: ”Varför håller ni med? Vad håller ni med om?” bjöd in eleverna att reflektera över andra elevers sätt att uttrycka element 7:s uppbyggnad. Att detta också skedde synliggjordes i såväl Sophies: ”På element 3, så är det tre där uppe och på element 5 är där fem där uppe, så på element 7 måste det vara sju.” som Gretes argumentation: ”För att på element 5 så / ... / var det ju fem / ... / och då blir det ju fem tvåor men på element sju blir det ju sju tvåor”. Båda eleverna tog i sina argumentationer på olika sätt stöd i relationen mellan elements position och antalet komponenter.

Diskussion

I föreliggande artikel adresseras frågan: *På vilka sätt kan didaktiska principer för lärandeverksamhet stödja ett etablerande och ett upprätthållande av ett algebraiskt arbete och därmed möjliggöra för ett urskiljande av kritiska aspekter?*

Centralt i etableringen av ett algebraiskt arbete är, ur ett lärandeverksamhetsperspektiv, att eleverna inledningsvis ställs inför en problemsituation som utmanar dem att engageras i ett algebraiskt (teoretiskt) arbete. Det är dock inte tillräckligt att det algebraiska arbetet etableras utan lika stor vikt behöver läggas på själva upprätthållandet av det. Den lärandeverksamhetsteoretiska principen om lärandemodeller har en central roll i såväl etablerandet som i upprätthållandet. Lärandemodellernas roll handlar dels om att möjliggöra ett kollektivt algebraiskt arbete dels att eleverna genom att kollektivt skapa, rekonstruera och bearbeta modellen får tillgång till sådana teoretiska aspekter av det kunskapsinnehåll som de ska lära sig (Eriksson, 2018; Gorbov & Chudinova, 2000). I detta learning study-projekt skulle det kunna tolkas i termer av att eleverna genom arbetet med lärandemodellen ges möjlighet att urskilja de kritiska aspekterna. Även den lärandeverksamhetsteoretiska principen motsättningar kan ha en tvådelad roll, det vill säga att fungera som en drivkraft i såväl etablerandet som i upprätthållandet av det algebraiska arbetet. Till exempel kan det handla om att läraren eller någon elev skapar ett matematiskt dilemma eller en hake som behöver hanteras av eleverna och som när det hanteras möjliggör för ett urskiljande av innehållsliga aspekter. Även här skulle det kunna uttryckas i termer av att motsättningen möjliggör ett urskiljande av kritiska aspekter.

I följande avsnitt kommer de lärandeverksamhetsteoretiska principerna som nämnts ovan att diskuteras i relation till skapandet av förutsättningar för ett algebraiskt arbete och hur eleverna därigenom ges möjlighet till att urskilja kritiska aspekter gällande förmågan att uttrycka och argumentera för algebraiska mönstergeneraliseringar.

Davydov (2008, jfr. Eriksson, 2017) beskriver hur aspekter av det teoretiska kunnandet elever förväntas utveckla behöver byggas in i de problemsituationer som elever ställs inför samt att dessa problemsituationer ska utmana elever till ett teoretiskt arbete i vilket de själva är aktiva agenter. Det kan beskrivas som att en problemsituation etablerades i forskningslektion 3 där eleverna utmanades till ett teoretiskt utforskande av mönstrets strukturella egenskaper. Problemsituationen etablerades genom uppgiften ”Rita element 4” tillsammans med lärarens uppmaning till eleverna att visa hur de hade tittat i de givna elementen för att rita figur 4. Det fanns inte uttalat i uppgiften eller i lärarens frågor att eleverna skulle jämföra givna elements strukturella egenskaper utan det var uppmaningen som inbjöd till detta genom att läraren bad eleverna visa hur de hade tittat i givna element för att förutsäga ett icke givet element. I förlängningen möjliggjorde även elevernas arbete med att jämföra givna element att de kunde urskilja den kritiska aspekten *relationen mellan elementets position och antalet komponenter.*

I relation till principen lärandemodell får de inringningar (se figur 9) som gjordes inledningsvis i den tredje forskningslektionen funktionen av en lärandemodell. Den skapades när eleverna skulle visa hur de hade tittat i givna element för att bestämma ett icke givet element. När det skedde fick även de andra eleverna tillgång till den kritiska aspekten *relationen mellan elementets position och antalet komponenter*. Detta möjliggjordes genom att grupperingar av komponenter i givna element kunde sättas i relation till elementets position, vilket därmed innebar att relationen visualiserades. Vidare fungerade inringningarna som redskap för klassrumskommunikationen och de reflektiva diskussionerna. Det skedde till exempel när läraren omvandlade inringningarna till tal felaktigt, och därmed utmanade eleverna till att argumentera kring varför och vad i omvandlingen som var felaktig. När läraren senare förändrade inringningarna i givet element och erbjöd en alternativ struktur blev det möjligt för eleverna att urskilja att *relationen mellan elementets position och antalet komponenter* kan uttryckas på olika sätt.

Även den didaktiska principen motsättningar i form av en hake visade sig ha betydelse för att eleverna skulle engageras sig i ett algebraiskt arbete. Resultatet i föreliggande artikel ger exempel på hur lärarens misstag att omvandla elevers inringningar i element felaktigt utmanade elever att ytterligare pröva och reflektera över relationen mellan elementets position och antalet komponenter. Det blev synligt genom att eleverna i sina argumentationer tog stöd i aspekter som till exempel *relationen mellan elementets position och antalet komponenter*. Vidare ges exempel på hur den planerade motsättningen mellan en lodrätt gruppering och en horisontell gruppering av komponenterna i givna element gjorde det möjligt för eleverna att utveckla en mer generell förståelse gällande hur *relationen mellan elements positioner och antalet komponenter* kan formuleras.

Denna artikel bygger på en begränsad kvalitativ studie, där tre forskningslektioner har genomförts. Trots detta kan den bidra till att fördjupa förståelsen gällande på vilka sätt principerna för lärandeverksamhet kan stödja ett etablerande och upprätthållande av ett algebraiskt arbete och därmed möjliggöra för elevers urskiljande av kritiska aspekterna. Resultatet kan användas som grund för fortsatta studier, exempelvis i en studie av en serie av lektioner där även mer komplexa mönster används.

Tack

Detta arbete finansieras och stöds av Stockholm stad, Forskarskolan i learning study samt Institutionen för matematikämnets och naturvetenskapsämnenas didaktik vid Stockholms universitet.

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Fotnoter

1 Begreppet ”algebraic activity” är ett begrepp som används av bl.a. Kieran (2008, 2011), Kieran, Pang, Schifter & Ng (2016) och Radford, Bardini & Sabena (2007) och har här översatts till algebraiskt arbete.

2 Bilden är efterkonstruerad utifrån Davydov, Gorbov, Mikulina och Saveleva (2012).

3 Learning study som forskningsansats möjliggör ett systematiskt utforskande och analyserande av vad som kan skapa förutsättningar i undervisningen empiriskt.

4 Valet att författaren genomförde forskningslektionerna var att ett nytt teoretiskt och didaktiskt ramverk skulle prövas och det fanns inte utrymme för de deltagande lärarna att sätta sig in i ramverket innan första forskningslektionen skulle genomföras.

5 Med forskningslektion avses här en lektion som designas, genomförs, analyseras och revideras i en iterativ process. Intentionen är att förfina undervisningsdesignen för en och samma lektion och genom det studera vad det är som skapar förutsättningar för elevers lärande gällande specifika kunskapsinnehåll. Det är med andra ord ”samma” lektion med samma innehåll som genomförs tre gånger med olika elever.

6 Bilden är efterkonstruerad utifrån Davydov, Gorbov, Mikulina och Saveleva (2012).

7 Lärandeverksamhet har under senare år använts som en alternativ lärandeteori till variationsteori i learning study (se t ex Björk m.fl, 2019; Eriksson, 2015; Eriksson, 2017).

8 Det övergripande forskningsprojektet som ligger till grund för denna artikel består av två delstudier, där delstudie 1 är en kartläggningsstudie gällande elevernas erfarande av mönstergeneraliseringar.

9 I utdragen har följande transkriptionsnycklar använts:
Kursivering innebär att läraren betonar ordet.
... innebär att läraren pausar.
Inom [ ] beskrivs gester och dylikt, det som uttrycks verbalt.
(---) innebär att det som sägs inte är hörbart.
Språket i utdragen har ändrats något, för att underlätta läsning. De redigeringar som gjorts har inte ändrat innebörden utan handlar till exempel om att halva ord har tagits bort.

10 Denna och liknande bilder är efterkonstruerade med stöd av videoinspelning.

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Abstract

The aim of the article is to describe and analyze what in different lesson sequences that creates the conditions for students to be involved in algebraic work and thereby distinguish critical aspects. The article is based on data from three research lessons in which *Learning activity* together with Radford's work on pattern generalizations were theoretical starting points. In the analysis, didactic principles of Learning activity along with a few identified critical aspects regarding the ability to express and justify algebraic generalizations served as analytical tools. The result can contribute to deepened understanding of the ways the principles can support the establishment and maintenance of algebraic work enabling students to distinguish critical aspects.

Chapter 3 – Exploring understandings of early algebra for citizenship

This chapter addresses the second research question in the thesis, namely how early algebraic ideas can be understood for the combined purpose of developing younger students' algebraic thinking and fostering citizenship. Exploring this question marked a turning point in my research trajectory, during which I engaged in the work of rethinking early algebra from a CME perspective considering current socio-ecological crises. The exploration started with a critical examination of key existing literature in early algebra with the purpose of identifying the dominant views on the purposes and concerns of early algebra research. Then I proceeded to broaden what school mathematics and early algebra with CME could be.

The chapter begins explaining the process of the critical review of salient overview literature in early algebra and presents Paper 3. Then the chapter unfolds the subsequent of taking on board a CME perspective in early algebra. In the final section, I articulate the justification for investigating new approaches to early algebra, thereby laying the groundwork for addressing the third research question.

Critically researching early algebra research

The aim of the analytical process in this part of my research work was to critically examine how the purposes and concerns of early algebra are articulated in research. Recognizing such articulations was an initial step toward connecting early algebraic thinking with efforts to empower students as future citizens, and as informing the development of possible innovative curricula and pedagogical approaches that challenge students' algebraic thinking. Table 2**Table 2** provides an overview of the work.

Table 2

Overview of work

Overarching research question	How can we understand powerful algebraic ideas for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens?
Empirical inquiry 3	Critically examine research on early algebra
Data	*Early algebraization: A global dialogue from multiple perspectives* (Cai & Knuth, 2011) *Teaching and learning algebraic thinking with 5- to 12-year-olds. The global evolution of an emerging field of research and practice* (Kieran, 2018a)
Analytical frameworks	Skovsmose and Valero's (2008) framework of *powerful mathematical ideas*. Pais & Valero's (2012) notion of *researching research*
Papers	Paper 3—Powerful algebraic ideas: Early algebraic thinking and active citizenship

In the following, I provide a detailed description of the data, the frameworks employed, and the three steps within the analytical process.

Description of data

The data examined were two major research review books in the field of early algebra: Cai and Knuth (2011) and Kieran (2018a). Cai and Knuth's book is part of a Springer book series, which presents a synthesis of widely cited research advances in mathematics education. This particular book highlights major studies and trends in early algebraic thinking globally, providing insights into its conceptualization and investigation. The authors examine how students develop algebraic thinking in elementary and middle school from curricular, cognitive, and instructional viewpoints, fostering international perspectives on early algebra. Kieran's book is based on ICME-13 Topic Study Group 10 on Early Algebra, with contributions from well-known scholars and up-and-coming researchers from around the world. It offers empirical findings from multiple countries and continents to explore how algebraic thinking develops and how it can be taught effectively from the earliest grades. These books form a collection in which scholars worldwide engage in a dialogue about early algebra ideas. They were, therefore, seen as providing a solid foundation for insights into algebraic thinking in a broader sense. In other words, they could serve as a basis for thinking of early algebraic ideas as efforts to empower students as future citizens.

Furthermore, it is important to note that my choice of data was based on identifying key algebraic ideas. I aimed to select ideas that could support expanding the concept of early algebraic thinking to include the development of active citizenship, rather than capturing the full scope of the field of early algebra research. Rather than providing a comprehensive overview of early algebra research, I aimed to explore how powerful algebraic ideas can be understood for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens.

The above also meant that I was not specifically interested in which author addressed which idea. The investigation, therefore, focused on the interpretation of "algebraic ideas" in the books in their totality. By adopting Pais and Valero's (2012) approach of "researching research" (see more below in the section: *Analytical* *framework*), the analysis emphasized the collective patterns and conceptualizations present in the reviewed books, rather than attributing specific ideas to individual contributors. This perspective enabled a more comprehensive examination of how algebraic ideas were interpreted across both works, providing a broader understanding of the field without limiting the findings to specific authors or studies.

Analytical frameworks and process

My analysis was informed by the framework of Skovsmose and Valero (2008). They sought to investigate how existing research in mathematics education implicitly or explicitly views mathematics education as powerful. They pursued three key questions: a) What is the understanding of "power" present in research? b) What is the source of the power of mathematical ideas? c) What are the consequences of that power? They identified four different interpretations of how mathematics education can empower students as future citizens:

- *Logically:* Mathematics is seen as a set of objects. The focus is on understanding mathematical concepts and doing mathematics for the sake of the internal characteristics of mathematics. The goal is to establish mathematical knowledge, understand its methods, and provide new insight into a different set of concepts. Being able to make abstractions is essential. Mathematics empowers students when they manage to acquire its internal logical characteristics.
- *Psychologically:* Mathematics is conceived primarily as a learning process. The focus is on capturing and facilitating the developmental nature of mathematical thinking toward higher levels of abstraction and formalization. The power of mathematics education lies in its developmental potentialities for the individual.
- *Culturally:* Mathematics is seen as a tool for relating to people's context and life conditions, for making decisions, for

participating in different practices, and for envisioning future life possibilities. Mathematics is powerful because it enables learners to understand and transform who they can become in their culture.

- *Sociologically:* Mathematics is seen as a descriptive and prescriptive resource and tool for action that formats society. Mathematics is powerful because it enables planning and decision-making to be integral components of technological actions. It also enables students to recognize the functioning of mathematics in society and how it connects to both its positive and harmful effects as a tool in technological action.

I sought to apply this framework to critically examine research on early algebra and to identify in which ways it conceives its knowing as powerful for students. Based on Skovsmose and Valero's framework presented above, I scrutinized the two early algebra review books, performing what Pais and Valero (2012) call "researching research". Rather than the typical exercise of reviewing literature to generate themes of categories of findings in research, Pais and Valero propose to identify the key statements that research produces to form the objects of which they speak. That is, the exercise of researching research allows us to make visible how expert knowledge actively constitutes particular ways of conceiving the research objects. By making this visible, the opportunity to find alternative views of such objects is made possible.

Step 1. Adjusting the framework proposed by Skovsmose and Valero (2008) to early algebra

I started producing a list of keywords relating each interpretation of powerful mathematical ideas to early algebra. These keywords served to identify segments in the two data sources that addressed any of the four powerful ideas:

- *Logically speaking*: abstraction, concept, generalization, mathematical structures, origin.
- *Psychologically speaking:* argumentation, mathematical thinking, mediating tools, modeling, reflection.
- *Culturally speaking*: critical decisions, cultural family, situated/situations, student's background, student's foreground.
- *Sociologically speaking*: agency, empowerment, citizenship, critical thinking, critical reflection, democratic, real, society.

Step 2: Identifying excerpts in the research review books

Undertaking a deductive analysis, grounded in Skovsmose and Valero (2008), I identified statements in the two books related to each of the four interpretations. I searched using the keywords above, and when I identified a keyword, I reviewed the text surrounding it to determine whether the excerpt aligned with the idea's intended interpretation. Throughout this process, the descriptions of the four powerful ideas were used to determine whether the excerpt corresponded to each description, as referenced by the established keyword. When a keyword was identified within the text, the entire segment containing it was highlighted in a designated color. For instance, the occurrence of the keyword "society," which served as a possible indicator of a sociologically powerful early algebra idea, led to that segment being marked yellow. This process proved challenging and in some instances it was necessary to expand the surrounding text, while in others, further review of Skovsmose and Valero's work was required to determine the appropriateness of the interpretation.

Throughout the assessment, the articulation of the four central concepts was used to determine whether the excerpt corresponded to their definitions, as referenced by the established keyword.

All the marked segments were then organized into four different documents, one document for each of the four interpretations of stressed early algebra ideas: logically speaking, psychologically speaking, culturally speaking, and sociologically speaking.

Step 3: Constructing descriptions of powerful algebraic ideas

Having compiled the segments into separate documents for each powerful idea, the primary purpose of Step 3 was to identify which early algebraic ideas were stressed, with a particular focus on those deemed culturally and sociologically powerful.

In each compiled document, I reviewed all segments to a) determine which aspects of early algebra were addressed, and b) assess their connection to the central, powerful idea. My identification of early algebra components focused solely on the specific algebraic concepts discussed, irrespective of context. To evaluate the connection, I referenced the previously summarized interpretations of the relevant powerful ideas.

To illustrate a) and b), I provide two examples related to excerpts 1, 2, and 3 in Table 3. Excerpt 1 pertains to sociological interpretation and Excerpts 2 and 3 to cultural interpretation. Excerpt 1 was identified as a sociological interpretation by the keyword "society," noted in yellow. The early algebraic aspect identified was "symbols (letters) to express relationships (to model) and thereby to resolve problems," noted in green. Because of its sociological emphasis, the algebraic aspects were identified as connecting to "Mathematics is seen as a descriptive and prescriptive resource and tool for action that formats society," within the sociological interpretation. The powerful algebraic idea—sociologically speaking—was then formulated as "Early algebra is powerful as a resource tool for action that formats society."

Excerpt 2 was identified as a cultural interpretation by the keyword "situations." This keyword refers to how mathematics can serve as a tool that aligns with people's contexts and life conditions—supporting decision-making, engagement in diverse practices, and envisioning future opportunities from the standpoint of a situated learner. The early algebraic aspect identified was "a cluster of modeling languages to express and support reasoning about situations being modeled" and "algebraic-linguistic aspects." Excerpt 3 was also identified as a cultural interpretation, but by the keyword "background." The early algebraic aspect identified was "algebraic-linguistic aspects." In this case, two excerpts from different parts of the review book were combined because they both addressed linguistic aspects of algebra. This combination was then formulated as one algebraic aspect, "Early algebra is powerful as a language to express and reason about situations being modeled." The algebraic aspects were then identified as connecting to "We interpret this dimension as emphasizing informal notions of algebraic concepts where language plays an important role" within the description of cultural interpretation. The powerful algebraic idea was then formulated as "Early algebra is powerful as a cluster of modeling languages both in and out of mathematics."

Table 3

An example of the analytical work in Inquiry 1

Excerpts	**Keywords**	**Identified aspects of early algebra and interpretation**
EXCERPT 1 However, it is probably universally agreed, certainly implicitly in the chapters of this section, that facility with the use of symbols (letters) to express relationships (to model) and thereby to resolve problems, is desirable if not essential for full participation in society and use of the power of mathematics. (C&K, 2011, p. 561).	Sociologically: Society	Early algebra: **symbols** (letters) to **express relationships** (to model) Sociologically speaking: Early algebra serves as a resource and tool for action that formats society.

EXCERPT 2 Algebra as the application of a cluster of modeling languages to express and support reasoning about situations being modeled. (C&K, 2011, p. 8)	Culturally: Situations	Early algebra: a **language** to express and reason about situations being modeled (excerpt 2), **algebraic-linguistic aspects** (excerpt 3) Culturally speaking: Early algebra serves as a cluster of modeling languages both in and out of mathematics.
EXCERPT 3 /…/approach to early algebra, their [students, our comment to clarify who “they” are] own background, the age of the students, the themes they want to deal with, their curiosity and so on. /…/ algebraic-linguistic aspects. /…/ it is necessary to build up an environment able to stimulate informally the autonomous elaboration of formal coding for sentences in natural language, discussing them with the whole class and gradually producing a playful, experimental and continuously re-defined appropriation of the new language. (C&K, 2011, p. 492)	Culturally: Background	

Note. The “Excerpts” column contains a range of citations sourced from two research review volumes. The “Keywords” column outlines the relevant terms referred to within an excerpt. The “Identified Aspects of Early Algebra and an Interpretation” column highlights identified algebraic aspects.

Following this, descriptions were constructed by using the markings and their interrelationships as a basis. In some cases, this process was relatively straightforward; however, in others, it presented greater challenges. Excerpt 1, in Table 3, illustrates an instance where the approach was uncomplicated, and then it connected powerful algebraic ideas, such as symbols to express relationships with full participation in society. There were also occasions when establishing a link between the interpretation and the excerpt was not feasible, leading to no description being provided. I also came to see, in certain instances, that different excerpts addressed the same algebraic idea. In such cases, I grouped the excerpts and linked them to the specific interpretation. This approach is illustrated in the above table, with respect to excerpts 2 and 3, which together contributed to formulating the powerful algebraic idea (culturally speaking) “Early algebra is powerful as a cluster of modeling languages both in and out of mathematics.”

III

Paper 3: Powerful algebraic ideas: Early algebraic thinking and active citizenship

Fred, J., Valero, P., & Van Steenbrugge, H. (2022). Powerful algebraic ideas: Early algebraic thinking and active citizenship. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.) *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*, (pp. 489–496). Free University of Bozen-Bolzano and ERME.

This paper addresses the following question: How can we understand powerful algebraic ideas for the dual purpose of developing students' emergent algebraic thinking and fostering future active citizens?

The findings comprise four interpretations of powerful algebraic ideas and are characterized logically, psychologically, culturally, and sociologically, as they appear in the two major review books on early algebra research. These four interpretations are not strict, clear-cut categories, but rather capture what can be foregrounded in each. The results discussed in this paper also address how, culturally and sociologically, a more critical take on powerful algebraic ideas appears to be scarce in early algebra research. This provides arguments that justify expanding early algebra to incorporate the socio-ecological.

Furthermore, the main result from screening two major research review books was that the logical and psychological interpretations of "powerful algebraic ideas" dominate. This result resonates with Skovsmose and Valero (2008), who, for the general field of mathematics education, observed a large number of publications adhering to these two interpretations of power.

The findings propose conceiving of algebraic thinking as a culturally and sociologically powerful tool for analyzing models that represent aspects of reality, thereby enhancing understanding of the limitations inherent in these models, which often depict only specific phenomena. Consequently, early algebra becomes powerful when investigating, interpreting, and reasoning about functional relationships between quantities within the framework of such constructed representations.

Powerful algebraic ideas: Early algebraic thinking and active citizenship

Jenny Fred, Paola Valero, Hendrik van Steenbrugge

Powerful algebraic ideas: Early algebraic thinking and active citizenship

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How can powerful algebraic ideas be understood for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens? To address this question, we have examined two major research review books on early algebra to investigate the interpretations of "powerful algebraic ideas" that are present in the books as a whole. Skovsmose and Valero's (2008) four interpretations of powerful mathematical ideas (which focus on the logical, psychological, cultural, and sociological power of mathematics) were used. We show that in the books and book chapters there is a dominance of the logical and psychological interpretations of the power of algebraic. Furthermore, the cultural and sociological interpretations appear connected to algebraic thinking as a resource or tool for action in "society". Advancing new possibilities of expanding the ways in which early algebraic thinking is made powerful for students is a challenge to research.

Keywords: Early algebra, algebraic thinking, powerful algebraic ideas, socio-political perspective.

A socio-political perspective on early algebra?

In the last decades, early algebra research has stressed the importance of students' introduction to algebraic practices in the lower grades to support students' emergent algebraic thinking (e.g. Kaput, 2008; Radford, 2014). Such an early introduction lays the ground for further algebraic thinking. Since algebra has a special position in mathematics, through its applicability in other areas and for its role in supporting general reasoning, conclusions, and proofs, an early introduction to algebraic practices and thinking is considered a foundation for realizing the intentions of the mathematics curriculum with respect to students' overall mathematical learning (Cai & Knuth, 2011). Simultaneously, interest on the socio-political dimensions of mathematics education has grown among researchers and practitioners (Gutiérrez, 2013; Planas & Valero, 2016). A socio-political approach to mathematics education considers the development of mathematical thinking and learning as an aim tightly connected with the overall societal intention of providing students with tools to become active citizens. In other words, mathematics education should offer clear opportunities to deploy mathematical thinking to consider and act on the problems and concerns of students as members of a society (e.g., Skovsmose, 1994). When bringing these two lines of research together, one could think that the aim of developing students' algebraic thinking is not only to advance the learning of further mathematics, but also more explicitly to empower children's critical reflection and democratic participation in communities of peers and in society (Hauge & Barwell, 2017). As it appears from the analysis of the basic works in our study, more often than not this combined aim is not so clearly articulated in research. Thus, bringing together students' emergent algebraic thinking and fostering future active citizens becomes a relevant challenge to advance a socio-political research work on early school algebra.

This paper has the intention of examining the way in which the aims of early algebra are expressed in research. Identifying how such aims are articulated is a first step in finding ways to connect them with the intentions of empowering students as future citizens, and in opening possibilities for new design of curricula and pedagogies that promote algebraic thinking.

We took inspiration in Skovsmose and Valero (2008), who reviewed how research in mathematics education views mathematics as being powerful and learning mathematics as empowering students and giving them access to participation in society and democracy. They reviewed research with three questions in mind: a) What is the understanding of "power" present in research? b) What is the source of the power of mathematical ideas? and c) What are the consequences of that power? As a result, they identified four distinct interpretations of the notion of "powerful mathematical ideas" that are present in the field. Mathematics and mathematics education are powerful in a *logical* sense when the focus is on the internal characteristics of mathematics. The learning of mathematics is then justified "for the sake of the internal characteristics of mathematics" (Skovsmose & Valero, 2008, p.15). Powerful mathematics education is powerful in a *psychological* sense when the focus is on the individual's possibility of acquiring mathematical knowledge. Powerful mathematical ideas are then defined in relation to the mental operations involved in the learning of mathematics, rather than the internal logics of algebra. Mathematics education is powerful in a *cultural* sense when the focus is on students' interpretations of their life possibilities in a context and reflects a situated learner's perspective, addressing students' background, but also their foreground, or how students perceive themselves as mathematicians in their future live. Mathematics education is powerful in a *sociological* sense when considering mathematics as a central tool for larger social action and organization. Powerful mathematical ideas, then, are "defined in relation to the extent to which they are used as a resource for action in society" (Skovsmose & Valero, 2008, p. 21).

In our case, these four interpretations invited us to inquire in which ways democratic access to early algebraic thinking can be considered as powerful for the purpose of connecting the development of algebraic thinking with the intention of education for an active citizenship. The question we want to discuss in this paper is: How can we understand powerful algebraic ideas for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens?

Methodology

To answer this question, we examined two major research review books in the field of early algebra (Cai & Knuth, 2011; Kieran, 2018) to investigate the interpretations of "powerful algebraic ideas" in the books as a whole. This process was done in three steps. First, an analytic tool was created (Table 1) where Skovsmose and Valero's (2008) three questions (a) What is the understanding of "power" present in research? b) What is the source of the power of mathematical ideas? c) What are the consequences of that power?) helped guiding the description of each interpretation. Second, we performed an exercise of *researching research* (Pais & Valero, 2012) on the two books to identify statements present in the texts related to each of the four interpretations. This was done by scanning for keywords associated to each interpretation, identifying excerpts that expressed these ideas, and articulating the overall sense of powerful algebraic ideas present in the books. In researching research as an analytical approach, we were not interested in pointing the particular author of an idea. Rather, we identified the regularities of what is being said about powerful algebraic ideas. Therefore, the excerpts that illustrate the interpretations are coded "C&K" and "K" plus the page number to point the source of the citations in Cai and Knuth (2011) and Kieran (2018) respectively.

Table 1: Summarised view of Skovsmose and Valero's (2008) Powerful mathematical ideas…

Logically speaking…	**Psychologically speaking…**
Mathematics is seen as objects. The focus is on understanding mathematical concepts and doing mathematics for the sake of the internal characteristics of mathematics. The goal is to establish mathematical knowledge, its ways of working and to provide new insight into a different set of concepts. Being able to make abstractions is essential. Mathematics empowers through peoples' enculturation in it.	Mathematics is conceived primarily as a learning process. The focus is on capturing and facilitating the developmental nature of mathematical thinking towards higher levels of abstraction and formalization. The mathematical power is situated in its developmental potentialities.
Culturally speaking…	**Sociologically speaking…**
Mathematics is seen as a tool to relate to people's context and life conditions, for making decisions, participating in different practices, and envisioning future life possibilities. Mathematics is powerful as it allows the understanding and transformation of who learners can become.	Mathematics is seen as a descriptive and prescriptive resource and tool for action that formats society. Mathematics is powerful as it allows planning and decision making as an integrated part of technological actions. It also allows to recognize the harms that mathematics, as a resource in technological action, can create.

Powerful algebraic ideas

Table 2 shows the approximate distribution of excerpts in the texts. From this distribution, it is clear that most of the terms appear in connection with the logical and psychological focus.

Table 2: Distribution of number of excerpts

Interpretation	Keywords	Nr. of excerpts
Logically speaking...	abstraction, concept, generalization, mathematical structures, origin	More than 200
Psychologically speaking…	argumentation, mathematical thinking, mediating tools, modelling, reflection,	More than 100
Culturally speaking…	critical decisions, culturally, family, student´s background, student´s foreground	Less than 20
Sociologically speaking…	agency, empowerment, citizenship, critical thinking, critical reflection, democratic, real, society	Less than 10

In what follows we characterize the four different interpretations of powerful algebraic ideas as they appear in the two books. We acknowledge that even though we try to keep the four interpretations as distinct perspectives and emphases, they are not strict, clear-cut categories. Rather there are fine lines of distinction and similarity between them. Therefore, we tried to capture what can be foregrounded within each interpretation.

… Logically speaking

Algebraic ideas are presented in the books as powerful logically speaking when stress is placed on the features of algebra such as abstraction, mathematical structures and generalization. This dimension of powerful algebraic ideas connects to the multiple perspectives one needs to understand concepts and by linking these concepts to one another. For example, generalizing is described as powerful in relation to processes of identifying mathematical structures and relationships in mathematical situations. Seeing and describing mathematical structures and relationships are also seen as powerful in relation to constructing meaning. For example, understanding the "multiple meanings of variables and the ability to employ variables to express mathematical relationships or situations" (K, p. 144) are also expressed in terms of powerful algebraic ideas. The power of thinking algebraically is described as empowering students to analyse relationships, to notice structures, to generalize, to problem-solve, to model, to justify, to prove, and to predict (K, p. 408). Furthermore, "structure in terms of an agreed list of properties" is seen as a powerful algebraic idea when using them as axioms for deducing other properties (K, p. 287). In sum, the power in this category is strongly related to the internal logics of algebra.

… Psychologically speaking

From a psychological point of view, powerful algebraic ideas seem primarily to be defined in relation to what students can grasp and give meaning to in the learning processes of algebra. For example, symbolic notions are emphasized as tools in relation to students' thinking regarding generalizations and relationships between variables: "in particular, to recognizing the varying nature of variables" (K, p. 276). The algebraic symbols are also emphasized as powerful to describe and reason with overall mathematical ideas (C&K, p. 19). Schematizing, discovering patterns, "to imagine and to express, to specialize and to generalize, to conjecture and to convince" is also mentioned in terms of being powerful algebraic ideas (K, p. 334). This in relation to recognise "situations as instances of a class of similar situations, which constitute a person's example space" (K, p 334). "Observing examples to find regularities, noticing structure and relationships, forming conjectures about the observations, and then proving and concluding general statements" was also mentioned in terms of being powerful (K, p. 356). Mental models are mentioned as ways of thinking about abstract concepts (e.g., balance for equivalence) and/or represent abstract concepts (e.g., physical balances, balance diagrams, balance language, equations as balance). Transforming processes, the recognition of mathematical ideas and the use of mathematical strategies and analytical tools were mentioned (C&K, p. 332). Abilities such as meaningful symbolic reasoning are seen as powerful because they prepare for the abstraction of more advanced concepts and thinking in later grades. (C&K, p. 14). In contrast to powerful mathematical ideas in a logical sense, mathematical ideas are powerful because they relate primarily to students' *learning* of algebra rather than to a predominant focus on the internal characteristics of algebra.

… Culturally speaking

We interpret this dimension as emphasizing informal notions of algebraic concepts where language plays an important role. Competencies as to going from informal notions to more formal ways of mathematical thinking are emphasized (K, p. 29). Algebra is described as a cluster of modelling languages both in and out of mathematics (C&K, p. 492). The importance of tasks for students' lives is also emphasized for students to engage and participate (C&K, p. 430). Argumentation competences is attributed a crucial role (C&K, p. 469) even though algebraic thinking not always is mentioned directly; rather, activities with a special focus on argumentation. Further, is algebraic thinking highlighted as creating marginalization of students in schools and society as well as "a gateway to academic and economic success" and in that sense is algebra seen as valuable for the students. Culturally speaking powerful algebraic ideas recognize the *situated perspective* of learning algebra and the culturally loaded meaning associated to learning algebra.

… Sociologically speaking

The role of algebraic thinking is sometimes backgrounded, and sometimes it is foregrounded. Furthermore, "the use of symbols (letters) to express relationships (to model) and thereby to resolve problems" are mentioned in terms of powerful algebraic ideas (C&K, p. 561). "Real" problems are used as starting points where transforming processes, reorganizing mathematical ideas, schematizing, discovering relations and patterns, symbolizing, using analytic tools, and refining existing models are deployed as tools for students to have agency over their life situations and to critically scrutinize their environment interpreted as powerful algebraic ideas (C&K, 2011 p. 332). Students' abilities to make sense of data is mentioned, this in relation to whether and how different quantities relate to each other. In relation to this seem tables, graphs, and the use of symbols to work as tools or models to invite the students to describe and reason about mathematics ideas (C&K, 2011 p. 19). Algebraic reasoning is also highlighted, this in terms of being a powerful algebraic idea (K, p.380). In some excerpts is algebraic thinking expressed powerful in relation to work as a "fluid domain of thinking", "habit of mind" and as a particular resource to be used and integrated in every topic (C&K, p.18).

Discussion and conclusions

As described earlier on, we set us the task of discussing how to understand powerful algebraic ideas for the combined purpose of developing students' emergent algebraic thinking and fostering future active citizens. That is, we try to move early algebra education towards a practice at intersection of fostering future democratic citizens and the emergence of algebraic thinking for future mathematical learning. The main result, in screening two major research review books, is that there is a dominance of the logical and psychological interpretations of "powerful algebraic ideas". This result resonates with Skovsmose and Valero (2008) who, for the general field of mathematics education, could see a large number of publications adhering to these two interpretations of power. To keep assuming that the logical and psychological interpretations will lead to an empowerment of students may be problematic. As we know, the connection between mathematical knowledge and competence and relevant problems that people face in their lives does not happen "naturally" (Hauge & Barwell, 2017). As society gets more complex and children face the challenges of sustainability and climate change, it becomes necessary to make explicit connections between algebraic capacities/thinking and the ways in which algebra plays a role in addressing such shaky situations.

Our analysis of powerful algebraic ideas culturally and sociologically speaking can guide us in the attempt to enlarge the idea of algebraic practices where students are involved in more socially relevant issues, as for example climate change and sustainability, and where algebraic thinking becomes a resource or a tool for concrete critical thinking and action. One way of inviting students to be involved in this kind of activities is to work with models of situations that go beyond so called realistic or semi-real references for a controlled problem. For example, we can challenge students to read or model "reality" and then by raising certain questions/issues implicit make algebraic thinking becoming an analytical tool in the exploration of those models. Further, asking questions/making statements that invite the students to becoming aware of models' potential of visualizing only certain "things" and leaving other things unnoticed and invite them to reflect on what kind of consequences that may have. In this kind of work, algebraic thinking can become a language tool/resource as well as an analytical resource/tool (see also Blanton 2008). Another way of inviting students to this kind of work is in relation to technological action, where algebraic thinking can work both as a tool to create technological solutions as well as enable the students to detect that all things that are created with mathematics are not all good. Thinking in this direction, the contextualization becomes crucial, this in the establishment of an arena where algebraic thinking can operate both as a source of power but at the same time inviting students to critical examinations of mathematics itself. This implies a need to uncover the contextualization of a certain structure and at the same time creating another kind of need for algebraic thinking.

The remaining question is then, which could be new possibilities to build those kinds of arenas with the combined aim for early algebraic thinking? We believe that as a community of researchers, we could embrace the challenge of designing algebraic practices that place algebraic thinking in a larger context. In contrast to the usual work with problems and contexts tailored for algebra, we see that *wicked problems* (Block et al., 2018; Hauge & Barwell, 2017; Jurdak, 2016; Rittel & Webbers, 1973; Steffensen, 2021) can be a fruitful idea to both empowering the contextualization as well as creating another kind of need for algebraic thinking. Wicked problems can help us bridging the gap between school algebra and more socially relevant issues (e.g., Jurdak, 2016; Steffensen, 2021). Wicked problems can be described as complex problems with no definite formulation of what the problem and its solution actually is. Then the problem-formulations are vague and involve different interests and/or perspectives which encourage to negotiate disagreements that open up for different framing of the problem. Thus, if we want to go beyond providing students with tools to solve problems encountered in real life and instead invite the students to work as mathematicians handling conflicting stakes, complexity, decisions, and uncertainty, exploring algebraic activity to address wicked problem can be rewarding.

Neither of the above, however, comes without challenges. There is a risk that the students' either disregard mathematical (in this case algebraic) aspects or disregard socio-political aspects or disassociate them. Our further research intends to explore this further by imagining wicked problems in an iterative and collaborative process including pre-service teachers, teachers, researchers, and teacher educators, where we also stage these imagined wicked problems with 6–9-year-old students.

Acknowledgment

This paper reports on research conducted within the graduate school *Relevancing mathematics and science education* (RelMaS), funded by the Swedish National Research Council.

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Chapter 4 – Orienting what and how to teach early algebra and the socio-ecological in research literature

In this chapter, I start addressing the third research question: What characterizes activities that integrate both algebraic and socio-ecological aspects? Integrating early algebra and socio-ecological critical situations involves more than simply introducing students to contextualized problem situations; it requires enabling the context to actively inform students' engagement with both early algebraic and socio-ecological aspects. Thus, the context serves not merely as a backdrop for algebraic work but as an intrinsic component of the learning process. Emphasis, therefore, shifts from a predominantly algebraic focus to an examination of how algebraic concepts interact with socio-ecological within a complex web of relationships. But how can we understand that in more concrete ways?

In the following, I turn to the existing research literature within the CME focusing on socio-ecological critical situations to advance my and others' understanding of how contexts, socio-ecological critical situations, and mathematical abstractions can meet inseparably. First, I elaborate on how the notion of learning model can be considered a possible sociological learning model when connecting it to socio-critical modeling and an expanded view of the function of diagrams and diagramming. Second, I focus on research literature that invites me to advance in connecting early algebra to socio-ecological critical situations through mathematical models. Thirdly, I elaborate on wicked problems as critical problem situations. I conclude this chapter by addressing the need to explore critical problem situations that connect early algebra and socio-ecological crises.

Learning models and socio-critical modeling

In my licentiate thesis, the notion of learning models from LAT enhanced the materialization of algebraic thinking in whole-class discussions and fostered a shared exploration of mathematical objects through the materialization of arguments, problems, and collective memories (see Paper 2, see also Eriksson et al., 2019, 2024). This allowed me to raise a didactic question about how learning models can also help materialize exploration and reflection of socio-ecological critical situations that encompass ethical, social, and ecological values. That is, how can learning models which support algebraic thinking "psychologically speaking"—using Skovsmose and Valero's (2008) terminology—become a tool in supporting powerful algebraic ideas "sociologically speaking"?

Coles's (2021) way of addressing the possibility of integrating Davydov's view of theoretical thinking with socio-ecological crises can be connected to the consideration of learning models as sociological tools, that is, it materializes structures that are of a socio-ecological nature. Coles emphasizes that the connection between theoretical thinking and socio-ecological critical situations lies in considering abstract structures, discerning relationships within complex systems, and representing these through symbols or models. Sociological learning models can then serve as tools to materialize exploration, identification, and reflection of abstract structures and relations that constitute socio-ecological critical crises.

Coles (2023b) and Solares et al. (2022) highlight the importance of *socio-critical modeling* for linking mathematics with socio-ecological crises. To illustrate this connection, Solares et al. (2022) describe a classroom investigation of animal biodiversity in students' local environments, in which students collected data and represented their findings using formats such as pictograms and tables. In this way, modeling functions as a means of exploring, analyzing, and understanding a socio-ecological issue within students' lived experiences at school, while simultaneously bringing mathematical ideas to bear on that issue. More broadly, this orientation aligns with Barwell et al. (2022), who argue that the meaning of mathematics extends beyond purely mathematical contexts and emerges in situations where socio-ecological and mathematical dimensions are addressed concurrently in students' work.

As an early contributor to the concept of socio-critical modeling, Barbosa (2006) supports this orientation and articulates why it can be viewed as a learning model:

> […] modeling as a learning milieu where students are invited to take a problem and investigate it concerning reality via mathematics (Barbosa, 2003). This notion is quite removed from the characterization of modeling as involving diagrammatic representations. It refers to modeling as a school activity, which a pragmatic, scientific, or socio-critical perspective may inform. (p. 294)

Barbosa emphasizes that modeling can function as a process through which students investigate a problem in relation to reality. Accordingly, the focus shifts from the acquisition of specific mathematical techniques to students' development as critical modelers who can recognize and question the power of modeling. On this basis, socio-critical modeling may be understood as aiming to foster students' critical understanding of the world through engagement with socio-ecological critical situations. Building on these insights, I argue that socio-critical modeling can inform the development of sociological learning models.

At this point, the question emerges of what other tools in mathematics can be used to serve the function of learning models in the broader sense of being powerful tools "sociologically speaking". De Freitas and colleagues (e.g., de Freitas 2012a, 2012b; de Freitas et al., 2022) propose that diagrams possess significant potential as abstract tools for spatial reasoning and for reconfiguring the relationships between humans and the Earth:

> We treat the diagram as a creative abstraction, a method for thinking about knotted interactions, rather than a reductive form of representation (de Freitas, 2012a, 2012b). […] Rather than representing relations, the diagram becomes a creative abstraction when it makes new realities possible by un-making previously determined claims. (de Freitas et al., 2022, p. 508)

From the perspective of de Freitas et al. (2022), the function of diagrams can be viewed as sociological learning models, serving both as abstract representations that foster creative thought, but also as important tools to construct alternative perspectives and new hypothetical scenarios. Coles (2023b, p. 28) further develops this idea by linking it to the "processes of creative abstraction," emphasizing that the model or diagram itself is not an endpoint. Rather, it is a tool for investigation and for making sense of the underlying abstract structures associated with socio-ecological issues.

In an interdisciplinary sustainability context linking mathematics with science and technology, Savard (2017) describes a teacher-built 3D model representing one cubic meter of waste as a tool for making scale and quantity tangible. In the Grade 5–6 activity "Nothing is lost," the model helped students estimate waste volume and mass across different time spans and supported connections between these measurements and related science and technology concepts, even when teachers did not articulate explicit mathematical learning goals. Read through the lens of learning models, Savard's example can be interpreted as an embodied learning model: understanding develops through physically situated engagement with a material representation, where measuring, lifting, comparing, and estimating make otherwise abstract quantities experientially accessible. This positions the 3D waste cube not only as a teaching aid but also as a mediating artifact that organizes learners' attention, action, and meaning-making around the concepts of volume and mass.

In sum, socio-critical modeling frames students' mathematical work as an inquiry into socio-ecological critical situations. De

Freitas' account of diagrams as creative abstractions adds a complementary emphasis on how such inquiries can be materially and conceptually reconfigured by introducing new relations and possibilities for action. Together, these perspectives can support the development of learning models as powerful mathematical tools in a sociological sense: learning models that materialize socio-ecological structures and relations.

Mathematical models, modeling, and socio-ecological situations

The previous discussion turns the attention to students' work with mathematical models, and how such models can be thought with respect to what they allow students to learn while integrating both mathematical activity and socio-ecological critical situations. As I engage in a discussion of existing research, I also highlight possible connections that may inspire work in classrooms, with teachers and students, bringing early algebra and the socio-ecological together.

Three views of mathematical models and crisis

Skovsmose (2021) gives other examples of how models of reality can be incorporated into teaching and learning practices to develop students' awareness of what these models do. He addresses three types of relationships between models, mathematics, and crises. One relationship concerns mathematics as picturing a crisis. This relationship addresses how mathematical models can represent pieces of reality, such as climate change crises. In this case, the focus is on representing reality using a mathematical model. In the modeling process, the first step concerns specifying the part of reality that becomes modeled, the second step concerns specifying the mathematical concepts that are applied (parameters, functions, equations, etc.), and the third step concerns figuring out how these entities—the mathematics and the feature of reality—become related within the mathematical model. An important question in this case becomes the extent to which a mathematical model provides an accurate picture of the selected piece of reality (Skovsmose, 2021). This process supports thinking about how the relationship between mathematical models and crises can serve as a problem situation for teaching. The context of socio-ecological crises can be integrated into students' work on creating mathematical models, in which the pieces of reality are socio-ecological in nature, and where early algebra is integrated in the second step, when deciding which mathematical concepts to apply. In the third step, simultaneous integration can be realized when reflecting on how mathematics and the features of reality relate to the mathematical model.

A second type of relationship is that mathematics itself constitutes a crisis. This relationship is illustrated by the fact that a mathematical model does not always operate independently of a crisis; it can play a central role in the situation. For example, algorithms may serve as driving forces that actually contribute to the emergence of a crisis. Skovsmose (2021) gives the following example:

> […] one can think of piloting an airplane. The automatic pilot can take over, but even when the real pilot is in charge, many estimations and maneuvers are made automatically. The steering by the real pilot is based on a huge number of mathematics based automatic processes. The degree of automation gets more and more profound, and any such automatisation is constituted by a configuration of mathematical algorithms. Like any such configuration, unexpected implications might occur. As with financial crashes, so also airplane crashes might have their explanation in some automatised mathematical deepstructuring machinery going awry. (Skovsmose, 2021, p. 376)

Skovsmose discusses the critical role mathematics plays in algorithms that drive automated processes. Rather than merely representing routine transactional operations, these algorithms execute the transactions themselves; thus, the mechanics of each transaction are shaped by underlying algorithms that remain unseen. This perspective encourages teachers to design learning experiences where students investigate such hidden algorithms in their own environments. In this context, the algorithm functions as an implicit model of reality. By challenging students to examine the socio-ecological values and abstract structures embedded within these algorithms, opportunities arise to integrate socio-ecological themes and early algebra into classroom activities.

A third type of relationship concerns mathematics as formatting crisis. This relationship addresses how mathematics formats both readings and actions, ultimately shaping the dynamics of the crisis itself. In this sense, a model of reality not only describes a situation or provides a prediction, it also forms our perception of the situation and, therefore, how we tend to act in it. This relationship supports thinking of how a teaching situation can invite students to explore the formatting power of mathematics in models of reality (see more in the section: *Taking on board a CME perspective in early algebra*). In this case, the contexts of socio-ecological issues and early algebra can be integrated into students' work when they are challenged to explore underlying socio-ecological values, such as carbon footprint and sustainability. These underlying abstract structures may exercise formatting power, and students can also be invited to reflect on them.

Modeling in critical situations

The research literature on mathematical models and modeling in teaching and learning mathematics is extensive (e.g., Krawitz et al., 2025). It is also known that there is an emphasis on the use of mathematical modeling at the secondary level, while its use at the primary level is limited (e.g., Wei et al., 2022). Indeed, Wei et al. (2022) identified 10 out of a total of 239 publications with a focus on primary level mathematical modeling. To the question of the effects of teaching involving modeling activities on primary school children, they conclude that "through mathematical modeling, students can generate mathematical ideas, explore mathematical theorems independently, develop critical thinking, and improve their metacognitive and communicative skills." (p. 923). Their result shows that the documented effects are mainly connected to children's mathematical knowledge and competence.

The question remains, of how the awareness on the different types of mathematical models, as proposed by Skovsmose (2021) above, and teaching using mathematical modeling activities may open to new possibilities that allow to connect in significant ways critical situations and students' life-worlds and their role as citizens. Therefore, in what follows I focus on research literature that explicitly articulates the significance of the modeling context to characteristics of teaching that may support my thinking on the integration of early algebra and socio-ecological aspects. I both present the research and reflect on its connection to my purpose.

Research has argued that contemporary eco-sociological challenges have increased public use and exposure to mathematical concepts. During the COVID-19 pandemic, for example, terms such as "mathematical model", "flatten the curve", "peak infection", and other mathematical terms were widely circulated across newspapers, radio, TV, and social media (Stephan et al., 2021). The use of graphs in public communication increased (Kwon et al., 2021), and statistics were more intensely mobilized as political rhetoric (Yoon et al., 2021). Thus, mathematical language has played a crucial part in public and personal discourses by describing and modeling current and potential future scenarios used to explain and justify actions, regulations, and restrictions imposed on society. What implications does this situation have for teaching and learning mathematics that bring mathematical modeling to the fore?

Kwon et al. (2021) have analyzed the use of graphs in the Korean news media during the COVID-19 pandemic, examining both the types and frequency of graphs in news stories. They identified instances in which the mathematical misuse of these graphs may bias readers. Based on these findings, they argue that there is an increasing need to incorporate the context, central to "beyond the graph" reasoning, as a focal point of graph education in school-level mathematics. They highlight the importance of incorporating contextual values, not just the mathematical value of quantity. Their methodology for analyzing graphs in news media also offers insights into what it means to go beyond the mathematical model. They also provide insight into how to think about mathematical models in teaching. In other words, ideas on how the interconnections among graphs and other mathematical models, early algebra, and socio-ecological issues can be used in teaching and learning in school mathematics. This offers perspectives for envisioning activities for a double purpose.

Steflitsch and Kollosche (2025) give another example in which graphs are integrated. Their example concerns how Grade 8 students are invited to connect descriptive statistics, graph interpretation, units, large numbers, and the socio-ecological issue of water consumption in food production. One result shows that the connection between mathematics and socio-ecological issues is not always apparent to students, although they report that the relevance of the social topics significantly contributed to their overall engagement. So, the question here becomes whether students learn mathematics in these situations, even if they do not see it themselves.

Similarly, Steffensen and Kacerja (2021) provide an example of using existing algorithm-based models—the Carbon Footprint Calculators (CFCs)—as starting points to invite students to reflect on climate change and the formatting power of mathematics. Throughout this process, students considered the implications and applications of using a CFC:

> […] how the different variables impact the result; attempted to give meaning to the CFC by breaking down its components; explored extreme values and compared the individual average; reflected on the people behind the CFC, on its hidden assumptions, and its limitations; reflected on the global, national and individual emission; and reflected on the "take-home" and "bring for-ward" message. The students reflected about some sides of the modeling process where the designers include certain offset variables, such as the decision to give a certain value to food types independent of their length of travel. (Steffensen & Karceja (2021, p. 521–522)

The preceding discussion highlights the fundamental abstract structures and implicit values inherent in the CFC model. When applied to a teaching situation, this example demonstrates how students can engage in exploratory work utilizing CFC as a modeling tool. In doing so, they develop an awareness of the influential role mathematics plays in structuring and interpreting complex problems.

Coles (2023b) presents forward-looking ideas relevant to teaching situations, models of reality within the mathematics curriculum, and the socio-ecological context. He illustrates these ideas by graphing rational functions, explicitly connecting the graphs to the relationship between social-ecological crises and time. Coles suggests that students' work may encompass various scenarios; for instance, teachers can pose questions to focus students' attention on how social-ecological factors relate to time across different graphical representations. Students may subsequently be tasked with generating graphs and formulating corresponding equations. A further step is to ask students to use their equations to make future predictions. This example demonstrates how a problem situation can be structured in multiple stages, beginning with an existing graph, and highlights how targeted teacher questioning can draw students' attention to socio-ecological dimensions. In this context, early algebra serves as an analytical tool for linking temporal and socio-ecological variables. The process encourages students to identify abstract mathematical structures both when constructing graphs from equations and deriving equations from graphs. As Coles emphasizes, the teacher's role is pivotal in ensuring that instructional focus extends beyond algebraic manipulation, providing space for socio-ecological considerations within student work. Incorporating problem situations in which students use these equations to make future-oriented predictions is one approach to integrating socio-ecological aspects with early algebra.

Coles (2023b) argues that decisions about what to teach can include scrutinizing how mathematical models and representations are used, for what purposes, and in whose interests. In a similar vein, Thanheiser et al. (2025) propose guiding students to interrogate the formatting, framing, and narration of data visualizations to make explicit the layers of implicit and explicit decisions involved in their construction, using prompts such as "What is measured and counted?", "How are measures defined?", and "How are the data generated?"

In relation to this, activities can direct attention to how mathematics, including algebra, functions as a "language" for modeling, describing, analyzing, and communicating socio-ecological problems, while also contributing to the creation of these problems (Coles, 2022). In other words, invite students to reflect on how mathematical language can shape decision-making and influence choices that align with a particular agenda.

The what to teach, therefore, also needs to extend beyond the use of early algebra to describe, predict, and communicate socio-ecological issues (e.g., Barwell, 2018). It also needs to address the power that algebra in action can perform. This framing implies that what to teach also includes the development of critical abilities, such as recognizing that mathematical models are not equivalent to truth (Skovsmose, 2021, 2021, 2024).

Skovsmose's (2023b) ideas of different relationships between mathematics and crises—mathematics can represent a crisis, constitute a crisis, or shape a crisis—is another example of what to teach can include (see more in the section: *Mathematical models and concepts*) (Skovsmose, 2021). This framework underscores the multifaceted nature of mathematics.

Accordingly, determining what to teach and learn in early algebra, such as for example choosing variables, deciding what counts as data, expressing relationships, and using those relationships to justify decisions about what should happen next, is intertwined with understanding the various ways in which mathematics interacts with crises. This highlights the formatting power of mathematics, which is explored further in the chapter "Taking on board a CME perspective in early algebra" (p. 34).

Wicked problems as critical problem situations

The licentiate thesis indicates that the type of problem situation students encounter influences their engagement in algebraic work. In considering a critical problem situation that invites students to engage in activities integrating algebraic and socio-ecological dimensions, I have reviewed the relevant research literature to further develop this line of inquiry.

In my reading of research literature, I encountered Block et al. (2019), who discuss teaching and learning in relation to wicked sustainability problems. Reading this, I could see a connection to critical socio-ecological situations and, by extension, to CME. In particular, Block et al., in their arguments for working with such problems with students, require leaving room for uncertainties, dealing with diversity and controversy, thereby opening up questions regarding what the right actions and good values are. Also, the entanglement of facts and values in disputes over sustainability issues can give rise to ethical reflection.

The concept of wicked problems was introduced in the early 1970s to describe complex, multidimensional predicaments that resist straightforward policy solutions and may not be solvable in any definitive sense (Rittel & Webber, 1973). Rittel and Webber contrasted such problems with "tame" problems typical of the natural sciences, arguing that many contemporary societal–technological–scientific challenges, such as sustainability and climate change, are inherently multilayered and value-laden. Rittel and Webber (1973, p. 161–167) characterize wicked problems by the following 10 points:

- There is no definitive formulation of a wicked problem
- Wicked problems have no stopping rule
- Solutions to wicked problems are not true-or-false, but good-or-bad
- There is no immediate and no ultimate test of a solution to a wicked problem
- Every solution to a wicked problem is a 'one-shot operation'; because there is no opportunity to learn by trial-and-error, every attempt counts significantly
- Every wicked problem is essentially unique
- Every wicked problem can be considered to be a symptom of another problem

- The existence of a discrepancy representing a wicked problem can be explained in numerous ways. The choice of explanation determines the nature of the problem's resolution
- The planner has no right to be wrong

Wicked problems can thus be described as complex problems with no clear formulation of the problem or its solution. The vagueness of problem formulations and the involvement of different interests and/or perspectives can encourage the negotiation of disagreements. This can open up different ways of framing the problem. Thus, if we want to go beyond the usual notion of problem solving in school mathematics, in which the aim is for students to reach a final "correct" solution (e.g., Marchant et al., 2022), and instead invite students to work with conflicting stakes, complexity, choice and uncertainty, to get inspiration from the notion of wicked problems to re-think algebraic problem situations can be a rewarding move.

Along these lines, algebraic 'tame problems' can be understood as many of the problem situations used in typical school mathematics word problems, where students do some algebra, such as solving an equation without context. In a tame problem, it is clear what the students should do, and it is also clear "whether or not the problems have been solved" (Rittel & Webber, 1973, p. 160). When working with wicked problems, it is neither clear to students what to act on nor what a solution might be, and personal and environmental aspects are also at stake beyond mathematics. Thus, working on such problems may involve both early algebraic and sociological framings. Students could be encouraged to negotiate disagreements and to be open to different ways of approaching and framing the problem. That is, wicked problems are fraught with dilemmas that need to be addressed as part of facing the problematic situations. In such contexts, early algebra can serve both as a means for investigating the issue and as a subject for further examination.

In education, the notion of a wicked problem has been used to design learning situations that extend beyond the closed, well-defined tasks common in school subjects related to science education (e.g., Lundegård & Caiman, 2019) and sustainability education as part of designing "post-normal education" (Block et al, 2018). The notion of wicked problems has also been used in some cases in mathematics education. For example, Chan et al. (2021) highlight wicked problems in the special issue of Educational Studies in Mathematics on the COVID-19 pandemic, noting that it offers an early example of addressing "inescapable, high-stakes human problems" (p. 2). Steffensen et al. (2018) connect the wicked problem of climate change with mathematics education to help students discuss causal relationships and make informed, reflective choices that support action toward a more sustainable future. Their approach also encourages students to develop their own responses and critically examine how mathematics is used in societal decision-making.

Kynigos (2024) provides an additional example by giving students a digital game in which they confront a wicked problem. Kynigos argues that the game models "wickedness" in a meaningful way and, in doing so, prompts players to make consequential, value-laden choices. These choices may involve considerations such as social hierarchies, institutional habits, emotional complexities, and the perceived risk of COVID-19 infection, and they invite students to reflect on how such considerations relate to their own values.

Integrating the socio-ecological with early algebra and moving beyond traditional problem-solving approaches in school mathematics—where students are typically expected to produce a single "correct" answer—can be enriched by engaging learners with conflict, complexity, choice, and uncertainty, as informed by the concept of wicked problems. Such an approach may provide valuable insights for reimagining algebraic activities.

A need to explore critical problem situations

The previous Chapter 3 and the present chapter have advanced my understanding of what may characterize activities that integrate algebraic and socio-ecological aspects. The CME perspective presented in Chapter 3, together with Inquiry 3, has strengthened the understanding of what critical problem situations may entail—for example, how powerful algebraic ideas operate culturally and sociologically. Early algebra is thus situated in students' life-worlds and shaped by ecological, social, political, and ethical dimensions rather than by neutral abstractions. Consequently, students need to encounter early algebra both as (i) abstraction that establishes particular ways of seeing structures and relationships and (ii) a means for revealing abstract structures and relationships in socio-ecological contexts. Students should also be given opportunities to reflect on how algebra is applied in real-world contexts and how it influences both our understanding of the world and the world itself. Finally, students should be encouraged to examine how stakeholders (e.g., politicians, corporations, and social media platforms) may use algebraic tools to address socio-ecological crises and societal challenges in ways that serve specific interests, and how such applications can shape public perception and responses to these crises.

The engagement with research literature, which this chapter addresses, has added further perspectives. First, it advanced my understanding of how early algebra can be connected to socio-ecological critical situations through mathematical modeling. This includes inviting students to examine what reality a model represents and to construct models that represent a selected reality. It also includes supporting students in analyzing the formatting power of mathematical models. In this context, the chapter highlights different forms of abstraction and their relationships, including Skovsmose's concepts of thinking, realized, and real abstractions. Second, the chapter indicates that sociological learning models may support the materialization of students' exploration of and reflection on socio-ecological critical situations, including ethical, social, and ecological values. Third, it invited me to view wicked problems may function as a rhetorical, creative, and critical/emancipatory design tool (Lönngren & van Poeck, 2021) for algebraic activities that position early algebra in a broader context. Compared with conventional early-algebra tasks and contexts, wicked problems can strengthen contextualization and create a different need for algebraic thinking. They also support the integration of CME, socio-ecological crises, and early school algebra.

However, these different advances require empirical investigation. This rationale underpins the fourth empirical study, which examines how the idea of AWP can be developed into a coherent conceptualization of this type of problem. The next chapter (Chapter 5) addresses this question.

Chapter 5 – Exploring critical problem situations for early algebra within the socio-ecological

This chapter addresses the steps taken to tackle the third research question: What characterizes activities that integrate both algebraic and socio-ecological aspects? It provides a methodological orientation to the empirical setup of my PhD work, which primarily focused on generating discussion and reflection on new ways of understanding algebraic activities.

The approach described in the sections below is the result of an analytical process that began with the research reported in papers 4, 5, and 6, which is why these papers are included in this chapter. The approach taken in these papers is then further developed and reported on in this chapter. It is this elaborated analysis that has helped identify and refine the features of AWPs, as presented in Chapter 6.

The structure of the present chapter is as follows: I begin by addressing the empirical setup and the formation of a research team. Then, I apply Skovsmose and Borba's (2004) model of critical mathematics education research to visualize the research process from a methodological perspective. I then present the data, the data processing, and the analytical work. Next come papers 4, 5, and 6, which formed the basis of the analytical approach, and I end this chapter with a description of the ethical considerations in doing this research.

An empirical setup

The empirical setup followed Atweh's (2004) idea of simultaneously developing practice—in this case, the practices of teaching and learning algebra incorporating CME—and the theory about that practice, namely the conceptualization of AWPs. This means that the understanding of early algebraic teaching and learning, including CME, as well as the conceptualization of AWPs that connect early algebra with current socio-ecological conditions, were not present at the beginning of this participatory exploration. Instead, these emerged from an interconnected, interactive process of collaborative imagination and exploration. I describe how a research team was built to facilitate this interconnected, interactive process.

Building a research team

The first step of the empirical setup was to build a research team comprising individuals with diverse backgrounds and experiences in mathematics education. This decision was grounded in the first methodological concern: generating broader thinking and conversations, thereby enabling participants to move beyond their usual boundaries and to engage together in challenging efforts. The choice was also based on the second methodological concern, which involved generating data to support the conceptualization of AWPs as a means of integrating early algebra with socio-ecological issues. These were concerns that had developed during my licenciate thesis, where collaboration with teachers made it possible to carry out the empirical studies (see more in the section: *Methodological orientation* in Chapter 2).

Building the research team took place during the COVID-19 pandemic. Given the prevailing physical-distancing restrictions, meeting in person with in-service and pre-service teachers proved difficult. Therefore, I contacted individuals in leading positions within schools—people I had previously worked with and those with experience in school-based research projects. I asked whether teachers in their schools might be interested in participating in a practice-based doctoral research project. I contacted three people who introduced me to potential participants: a school lecturer, an assistant principal, and a principal. These three people referred me to six potential participating teachers, of whom eventually four, all teaching at the same school, decided to join the research team.

Another justification for creating a research team of this kind is the experience I gained as a teacher in a school setting, where I was involved in a practice-based research project (Fred & Stjernlöf, 2014). I had also served as a local project leader and research assistant on a project funded by the Swedish Institute for Educational Research (Eriksson et al., 2019; 2021). From this perspective, it was also evident that school leaders or school lecturers (school lecturers hold a licentiate or doctoral degree and work as lecturers in schools) play a pivotal role in creating an enabling environment for teachers to engage in similar initiatives. Having a positive relationship with the school or other individuals responsible for teachers' pedagogical matters was likely beneficial, as it could facilitate more constructive dialogue.

To engage with pre-service teachers, I recorded a YouTube clip in which I introduced myself, shared my ideas about the project, and explained what their participation could offer them in their future studies in teacher education and in their forthcoming careers as primary school teachers. I also informed them that they would receive a certificate for participating in a practice-based doctoral research project, which they could later use when applying for a primary school teacher job. An email was sent to 74 students enrolled in the Teacher Education Program for Pre-School and Primary School Years (K–3), in which algebra is one of the mathematical content areas. Five students expressed interest in participating. I invited them to a Zoom meeting, where I presented more details about the project and their participation, and they had the opportunity to ask questions. Following this meeting, four students expressed interest in joining the research team. Two of them actively participated throughout the project, while the other two participated in specific parts of it.

Although the research team also consisted of my supervisors, Paola Valero and Hendrik Van Steenbrugge, they never met the in-service and pre-service teachers. Therefore, the research team consisted of two groups: the inquiry group, which included the teachers, pre-service teachers, and me; and the supervisor group, which included my supervisors and me. Even though only participants in each group were present in person, the discussions, including reflections, were also influenced by those who were not present, since I served as the bridge between the two groups.

To structure the work in the research team as well as to structure the empirical work, I drew on Skovsmose and Borba's (2004) model of critical mathematics education research. In the next section, I utilize this model to visualize how it supported the organization of the research team's collaborative work.

Visualizing the research process

The framework offered by Skovsmose and Borba (2004; Figure 3 next page) provided theoretical and methodological tools to guide the research team's work in envisioning and conceptualizing AWPs. Skovsmose and Borba note that an important activity in critical mathematics education research is to examine situations that do not exist but may be considered desirable and can come to life through collaboration. The model makes explicit three types of *situations* and the characteristics and qualities of the *processes* connecting them. The model's different notions support visualizing turns within the movements of investigating and reimaging the practices of teaching and learning early algebra.

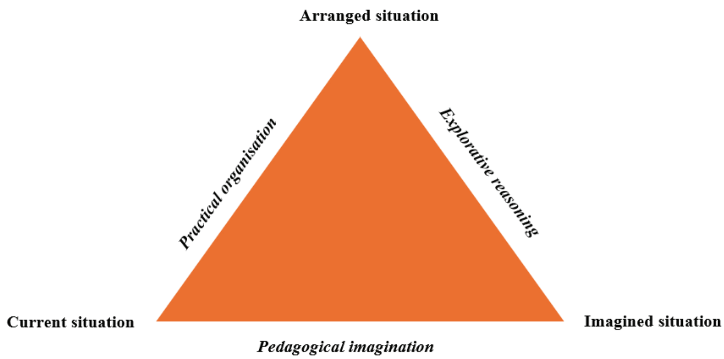


Figure 3 Skovsmose and Borba's (2004) model of critical mathematics education research

The *current*, *arranged*, and *imagined situations* help to identify how to move forward toward the conceptualization of AWPs. The *pedagogical imagination*, *practical organization*, and *exploratory reasoning* help characterize the processes that facilitate moving forward.

The *current situation* is the situation prior to the imagination of new possibilities (Skovsmose & Borba, 2004). It is "things as they usually are." At the beginning of my doctoral work, it addressed the common concerns and purposes of teaching and learning early algebra to prepare children for more advanced mathematics. The current situation drew on my licentiate thesis, and included algebraic activities, such as exploring, identifying, and reasoning about abstract structures in number patterns. It also included theoretical concepts derived from LAT, problem situations, contradictions, and learning models, all of which facilitate the establishment and maintenance of children's algebraic work (see more details in Chapter 2 and Paper 2).

The *imagined situation* envisions alternatives and new possibilities, drawing inspiration from different theoretical sources (Skovsmose & Borba, 2004). It refers to incorporating CME and the socio-ecological into the teaching and learning of algebra. The notion of imagined situation served as a tool for transcending the established narratives of early algebraic teaching and learning, enabling the imagination of something different. The imagination developed as the notions of CME and wicked problems fertilized ideas of early algebra based on LAT. In this context, Rittel and Weber's (1973) concept of wicked problem served as an orienting idea. Lönngren and van Poeck (2021) have pointed out that the notion of wicked problems in education has served as a rhetorical, creative, and critical/emancipatory tool. In imagining AWPs, the concept accomplished several essential functions. It structured communication and argumentation (rhetorical function, unfolded through exploratory reasoning), it facilitated the examination of alternative frameworks and interpretations (creative function, unfolded through the pedagogical imagination), and it promoted critical analysis and the transformation of established perspectives (critical/emancipatory function, unfolded in the practical organization). AWP were shaped not as predetermined or static scenarios, but rather as those designed to be implemented. AWPs offer ways to imagine alternatives to early algebraic practices.

The *arranged situation* serves as a practical alternative to the imagined situation, structured specifically to address practical or organizational limitations (Skovsmose & Borba, 2004). In this case, it refers to the space where an envisioned yet vague imagined situation was realized. It served as a space where we collectively formulated various possible AWPs, which pre-service teachers then tried out with small groups of K–3 students. Teachers also conducted some trials in their classrooms.

The *pedagogical imagination* refers to the characteristics and qualities of the process of moving forward in conceptualizing AWPs, where inventing alternatives and imagining are fundamental. However, this process of collaborative imagination did not happen simultaneously for both groups. One space involved the inquiry group, the in-service teachers, the pre-service teachers, and me, a teacher educator and doctoral student. The other space involved the supervisor group—me and my supervisors. In my role as an initiator and bridge between the groups, I mobilized conversations to bring ideas together and inventively propose new formulations of what early algebra can be when enriched with CME and the socio-ecological.

The *practical organization* refers to the process of designing and planning how the articulated possible didactical features of AWPs could be tried out with students. That means those features were used to design a concrete AWP, which were then tested with students.

The *explorative reasoning* refers to the process by which the articulated possible features were analyzed and reconsidered in light of how those features were used in the designed AWP, along with experiences from the tryout with students. The explorative reasoning differed in the inquiry group and the supervisor group. While the former focused on discussing and reflecting on the possibilities in practice, the latter contributed to further theorization and research.

Data production

The data production process became complex because it was initially unclear what could be considered as data. In this process, Skovsmose and Borba (2004) served as an inspiration to consider data not only in terms of the arranged situation and practical organization, but also in relation to the current and imagined situations, and the inquiry group's exploratory reasoning and pedagogical imagination. This means that, in this particular case, what counts as data includes material from all six situations and processes in the model in Figure 3.

The dataset was produced between February 2021 and May 2022. It included recordings of inquiry group meetings, the devised AWPs, and field notes and pictures of student work. All meetings within the inquiry group were video recorded using Zoom for online meetings or a video camera for in-person meetings. These recordings were transcribed. The inquiry group meeting data comprise the process of designing AWPs (*pedagogical imagination)*, working out lesson activities (*practical organization)*, and reflecting on in-class tryouts (*exploratory reasoning)*. While the major data corpus relates to the transcribed inquiry group meetings, the data additionally also include the AWPs formulated by the inquiry group and enacted in classroom settings, as well as field notes from the *arranged situations*; that is, tryouts, in which students worked on these problems. Below is an overview of the data.

Table 4

Dataset generated in the empirical setup

Data
10 video-recorded and transcribed inquiry group meetings (which lasted on average 52 minutes)
3 formulated concrete AWPs enacted in classroom settings
21 pictures of students' production when working with AWPs from the different tryouts (without faces)
6 pages of field notes from 3 tryouts of 3 concrete AWPs

The data analysis is reported in the sections below.

Data analysis

Data analysis entailed two stages. The first stage identified potentially relevant data segments from the collected dataset, listed in Table 4. In the second stage, the selected segments were analyzed in depth to conceptualize features of AWPs. In general, the identification of segments and their analysis drew on qualitative content analysis (e.g., Mayring, 2015), which aims at identifying recurrent patterns in what is expressed in large amounts of text. The selection and analysis had an abductive character, in that it moved back and forth between concepts and ideas available at each stage and the datasets (e.g., Alvesson & Sköldberg, 2018). I describe the two stages below.

Stage 1. Selecting relevant data segments

The analysis began with identifying segments of the transcribed inquiry group meetings as the basis for analysis[4]. Relevant data segments were selected using MAXQDA. MAXQDA was used to organize the transcripts and to mark and select segments according to selection criteria. Five inclusion criteria were defined through repeated readings of the transcripts with respect to Research Question 3: What characterizes activities that integrate both algebraic and socio-ecological aspects? The five inclusion criteria and the rationale behind them are listed below:

- *Algebra*: includes segments that speak to early algebraic content, discussions about what algebra comprises, and about the usefulness of algebra. This code helps identify segments in the data that relate to the "algebraic aspects" of Research Question 3.
- *Social*: includes segments where different aspects of citizenship, such as students as active citizens, depending on age, experience, life-worlds are mentioned. This code helps to identify segments in the data that connect to the "socio-ecological aspects" in Research Question 3.
- *Climate*: includes aspects such as environmental issues, climate change, sustainability, and human impacts on non-human nature – connects to the "socio-ecological aspects" in Research Question 3.
- *Didactics*: includes issues pertaining to the organization of teaching AWPs, designing AWP activities – connects to "activities" in Research Question 3.

[4] The other sources of data in Table 1 were brought into the analysis in Stage 2.

- *Context*: stood out as an important feature that served to connect both algebraic and socio-ecological aspects – connects to the "integration" in Research Question 3.

While some criteria overlapped and certain criteria shared boundaries, such as context with social, climate, and algebra, each criterion retained unique significance within the data. For instance, context could pertain to social, climate, or algebraic aspects; however, in the transcripts, these elements were discussed independently of context. Consequently, it was essential to maintain these criteria as separate categories. The selected segments met at least one of the inclusion criteria. Thus, the selected segments represent specific sections of the transcripts in which one or more members of the research team discussed matters relevant to the inclusion criteria. In cases where a segment pertained to multiple criteria, all relevant criteria were applied.

Below, I explain the significance of each inclusion criterion used, presenting them individually. I also provide examples from the transcripts for each criterion; these segments have been translated from Swedish into English. For each segment, I include a justification for why it has been assigned that criterion.

Table 5

Illustrating the application of inclusion criteria with example segments

Criterion	**Segment**	**Justification**
Algebra	R: Because algebraic thinking should still be included... how does it relate to younger ages? It's really about relationships, connections, and structures, and that's what you give examples of when you reason. When you... that's what you're really saying, even	Algebraic thinking is addressed, emphasizing its focus on relationships, connections, and structures. The segment also notes that the concept can be challenging to

	though you don't... but as I said, we're working on it... there's no right answer. We don't really know, we're just trying to become wiser about... and I think we have become that. I feel wiser.	define or fully comprehend.
Social	R: /…/ then we have narrowed it down to some extent to the climate footprint. And if you then look at what abilities a democratic citizen has? Yes, it's really about arguing, reasoning, reflecting, critically examining, exploring, and looking for explanations and qualities in something. And I think... what it says there... and dealing with conflicting opinions and being an active agent. That you don't just take everything someone else says at face value, but that you also... /.../ act yourself.	This addresses different aspects that relate to critical citizenship, such as critically examining and exploring, seeking explanations, dealing with conflicting opinions, and being an active agent.
Climate	TS3: Yes. But no, it becomes interesting if you add an economic aspect as well. I think that's where what you were trying to say about cheap plastic comes in. That environmentally friendly innovation is very expensive. A Tesla is very expensive compared to a regular car, but it is more environmentally friendly. So, there may be another discussion about cheap consumption versus environmental friendliness, and so on.	The focus is on environmental issues across different aspects; for example, things that cost more to buy are more environmentally friendly when used. This also speaks to the abstract structures of a car's value and could therefore also be related to algebra.
Didactics	T2: I understand what you mean. You first have to make a list and pretend that Santa Claus is coming, and make a list of what you	Here, the teacher discusses strategies for setting up a problem situation, in relation

	want. First... say five toys and then the next that you explain it with an environmental footprint, and you explain that you get one [toy or thing], it costs the environment so much, and it may be that the funniest toy has the biggest footprint really... in the worst way for the environment, so that they have to re-evaluate really.	to what comes first and what comes next.
Context	R: You are used to thinking about mathematics, but when you then have to bring in this other thing and that it is not just about putting it in a context, it is not just that we should put it in a context, that they are e.g. environmental agents or something... but it should be some real problem that anyway... where they benefit from mathematics...	The focus is on contexts and elaborates on broader interpretations of contexts. Additionally, it emphasizes that encouraging students to act as environmental agents is insufficient unless mathematics is also integrated as a crucial component.

After this first round of selecting potentially interesting segments, I performed a second round of selection, allowing me to create more focus. More particularly, I used MAXQDA to identify segments that *simultaneously* related to three of the five inclusion criteria, resulting in the following three categories:

A) Social, algebra, and didactics
B) Climate, algebra, and didactics
C) Context, algebra, and didactics.

This decision aimed to deepen understanding of the possible links between algebraic and socio-ecological components for effective integration. Algebra and didactics were treated as constants within each category due to their fundamental importance to the inquiry, while social, climate, and context were treated individually to represent different dimensions relevant to socio-ecological considerations. Presented below is an example illustrating the intersection of climate, algebra, and didactics. I also provide a justification for selecting this specific segment.

Table 6

Illustrating Category A. Climate, algebra, and didactics

Segment	Justification
T3: But it could also be that you discuss with the students what they think this is worth, and then the students get to reason about the knowledge they have. I think they themselves could conclude that this mobile phone is not very environmentally friendly and expensive, as it costs quite a lot of carbon dioxide. /…/ T2: But is it true that something that costs a lot also costs a lot for the environment? It can actually be very cheap plastic; cheap toys do have an impact, not always, but they have more of an impact on the environment, depending on how they are produced and what materials are used. So there doesn't have to be a linear relationship between expensive toys and the environment, and the environment having the most impact.	Climate: The segment addresses issues such as being environmentally friendly, carbon dioxide, the relationship between what material is used and its impact on the environment. Algebra: The segment addresses algebra through the discussion of linear relationships. Didactics: The segment involves discussions concerning how to invite students to reason about the relationship between a mobile phone, its carbon footprint, and it being environmentally friendly or not.

The combination of the criteria climate, algebra, and didactics was justified because the related segments addressed teaching that included all three. For example, both algebra and climate are addressed simultaneously at the end of the segment: "So there doesn't have to be a linear relationship between expensive toys and the environment and the environment having the most impact." At the same time, the section can be linked to the criterion "didactics" because it begins by inviting students to discuss and reason about the topic.

The following table summarizes the total number of identified segments in each category, illustrating the distribution and focus areas of the empirical work.

Table 7

The number of segments in each category

Category of segments	**A: *Social,* algebra and didactics**	**B: *Climate*, algebra and didactics**	**C: *Context*, algebra and didactics**
Number of segments	20	36	3

These 59 segments across categories A, B, and C in Table 7 were central to my data, which I analyzed in depth in the next stage. Importantly, I also complemented the data selected in the segments with connected information from the overall dataset such as the AWPs formulated and enacted in classroom settings, photographs of students' work on AWPs, and field notes from the classroom tryouts of these AWPs, as shown in Table 4.

In some cases, these different data sources were connected and therefore treated as a single data segment. For example, observation notes from classroom tryouts could be connected to the concrete AWPs implemented and/or to photographs of students' work on AWPs. In these cases, the different data sources were therefore combined into a single data segment. For example, in Table 8, the data sources, observation notes, and the enacted AWP cannot be separated, as they build on one another. The teacher's questions (registered in the field notes) are directed to the table within an AWP (registered in the concrete AWP implemented). The students' responses to the teacher's question (registered in the field notes) use the table within the AWPs (registered in the concrete AWP implemented).

Table 8

An example of how three different data sources are counted as a single data segment

Data sources	Example
Field notes	The teacher asks: "Does the table feel fair?"
Enacted AWP	
Field notes	Student 4: That little LOL doll cannot be much more than LEGO, can it? Student 5: And then the scooter… Scooters are big…

	Student 6: But you can have a scooter for a really long time… And there are a lot of other things than just plastic.

Stage 2. Characterizing data segments and conceptualizing features of AWPs

In this stage, I examined the selected data segments through the lens of ten codes, which helped characterize them in greater depth and ultimately to conceptualize features of AWPs. This qualitative content analysis was theory-informed, using CME and LAT as analytical lenses to interpret the data. More particularly, selected concepts from CME and LAT served as codes and were grouped into an analytical framework that helped identify features of AWPs. The abductive character (e.g., Alvesson & Sköldberg, 2018) of the analysis allowed the development of an interpretation that connected the data segments and the central concepts from CME and LAT.

This stage of the analysis included two steps: the first characterized the data segments, and the second conceptualized the features of AWPs. I describe both steps in the two sections below.

Step 1: Characterizing data segments

During step 1, the focus was on a) analyzing the selected relevant data segments in terms of selected codes, and b) providing a description of how each segment spoke to the assigned codes. I employed ten central concepts as codes, selected for their significance in creating conditions for the simultaneous integration of early algebraic and socio-ecological aspects in students' work with AWPs (from now on, I refer to these central concepts as codes). By examining data segments through the lens of these ten codes and subsequently describing the segments based on the codes, I aimed to gain deeper insight into the conceptualization of potential AWPs features.

The codes of the analytical framework were selected because they foreground analytically distinct dimensions related to early algebra, CME, and LAT, thereby supporting the identification of salient features of AWPs and, in turn, the conceptualization of AWP. Code 1 concerns powerful early algebraic ideas identified in the analysis reported in Paper 3. Codes 2–6 address key concepts from CME (see more in the section: *Taking on board a CME perspective in early algebra*). Codes 7–9 address concepts from LAT (see more in the sections on theoretical and methodological orientations in Chapter 2). The ten codes are listed below, along with a brief rationale for their relevance in characterizing the selected data segments.

Codes

1. *Algebraic ideas, as culturally and sociologically powerful,* involve early algebra as a cluster of modeling languages, both in and out of mathematics, comprising symbols (such as letters), variables, and unknowns. It directs attention to how these elements can be used to express, reason about, and model underlying structures and relationships. Furthermore, it relates to how schematization, visualization, symbolic representation, and modeling languages may facilitate the identification of underlying structures, patterns, and relations.

2. *Thinking abstraction* refers to how a mathematical concept or model offers a specific representation of reality or phenomena, capturing a selected subset of its features. This code directed attention to the underlying structures that support values in a mathematical model. It thus concerns variables and parameters that define the values in a model or are used in an algorithm. It also highlights the fact that variables and parameters can encompass socio-ecological values.

3. *Realized abstraction* is connected to thinking abstraction when such abstractions are applied in practice, and we start

operating with them. That is, the perception of reality is shaped by features chosen for inclusion in a mathematical model. This code highlights how students may identify possible values, underlying structures, and their relationships, as well as key socio-ecological dimensions.

4. *Citizenship* refers to the role of mathematics within society and emphasizes the importance of fostering active, critical citizens. Citizenship points to reflection on values of an ethical, political, social, and ecological character. This code emphasizes contextualization, encouraging students to engage in critical reflection, empowering them as future citizens, and concurrently fostering their algebraic thinking. The code of citizenship directs abstract thinking toward societal considerations.

5. *Life-worlds* refer to the everyday relations and experiences students inhabit, in which different values—such as ethical, ecological, political, or cultural—are situated. In essence, life-worlds focus attention on mathematical concepts and models relevant to daily life, as well as on values that extend beyond purely quantitative considerations. This code directed attention to which socio-ecological objects within students' life-worlds may serve as topics for mathematical modeling.

6. *Critical situations* are those in which mathematics becomes real, affecting people in concrete, serious ways. This code helped identify data segments that drew attention to socio-ecological situations affecting people. It also supported a focus on simultaneously connecting early algebra and socio-ecological aspects, where underlying structures are of such a nature that they can affect people in a concrete and serious way.

7. *The formatting power of mathematics* refers to the establishment of connections between mathematics in action and issues related to power, abstraction, and socio-ecological concerns. This code was directed at the simultaneous integration of early algebra and socio-ecological aspects, framing early algebra as a tool for recognizing mathematics' formatting power in action and its orientation toward socio-ecological issues.

8. *Problem situations* foregrounded the contextualization of a problem situation that invites students to integrate both early algebraic and socio-ecological aspects.

9. *Contradiction* refers to dilemmas, conflicts, or situations that are seemingly incompatible in some dimension. This code highlights how contradictions can challenge students to explore, interpret, analyze, and reason algebraically about the underlying structures and potential relationships between given values or data. It also draws attention to socio-ecological contradictions.

10. *Learning model* refers to schematic, psychological, and sociological tools that materialize students' reasoning, exploration, and reflection, while supporting the integrated consideration of early algebraic and socio-ecological aspects. It refers to mediating tools in collective exploration and reflection. These tools were material, iconic, or semiotic in nature and can take various forms such as diagrams, sketches, or symbols. This code draw attention to what can be seen as the materialization of students' relating and structuring in action—for example, the materialization of students' reasoning, exploration, and reflection on the relationships among the three abstractions and on the formatting power of mathematics.

Having identified the ten codes, I reviewed the selected data segments. When I identified a data segment that aligned with one of the codes, I coded it accordingly. In some cases, the same data segment was assigned multiple codes.

Subsequently, the focus was on describing and explaining the connections I perceived between each data segment and its assigned code(s). The reason for creating these descriptions was that I anticipated that in Step 2, they could be used to identify and conceptualize potential features of AWPs. The codes added analytically distinct dimensions of early algebraic and socio-ecological aspects, as well as didactic principles that may support such integration.

In describing these connections, I used the codes' rationale for relevance, as listed above, to articulate what I observed in the data segment. By doing so, I was able to articulate a concrete connection between the data segment and its assigned codes, which then served as a description of what characterized the data segment. I describe this process using three examples below.

Example 1: Segment coded as life-worlds

The data segment from one of the transcribed inquiry group meetings in Figure 4 below was assigned the code *life-worlds*. The perceived connection between the data segment and the code lies in the socio-ecological issue of carbon dioxide, which concerns the everyday relations and experiences students inhabit in their life-worlds. This topic is addressed on social media, in families, and in schools in Sweden, so it is a topic that students can relate to.

> T4: Children know a lot, have heard quite a lot, and seen this thing about carbon dioxide and what it means. We can only use this much [carbon dioxide]… so they can certainly relate to that.

Figure 4 Data segment from transcribed inquiry group meetings coded for life-worlds

Example 2: Data segment coded for citizenship and critical situations

The data segment in Figure 5 below, drawn from field notes of concrete AWP tryouts, was assigned the codes *critical situation* and *citizenship* because carbon dioxide emissions affect people in serious ways. The perceived connection to citizenship was that the student reflected on the value of ethical character by expressing awareness of the relationship between consumption and our planet's resources.

> Another student explains that it is about how many resources are used and that this is proportional to the Earth's size. The same student wonders if it is true that the teacher takes up so much of the Earth's surface. The teacher refers again to the WWF calculator. The student expresses their shock at how many resources the teacher is using. And keeps repeating 4.2!!! Another student asks, 4.2 Earths? And asks if the teacher is destroying the Earth.

Figure 5 Data segment from field notes coded for citizenship and critical situations

Example 3: Data segment coded with thinking abstraction, realized abstraction, and the formatting power of mathematics

The data segment in Figure 6 below, drawn from field notes on a tryout of a concrete AWP and a formulated AWP, was assigned the codes *thinking about abstraction*, *realized abstraction*, and *the formatting power of mathematics*. The perceived connection lies in how students, in response to the teacher's question, engaged in exploration, interpretation, and reasoning to determine which object characteristics influenced the values assigned in the carbon footprint table. This process involved a *thinking abstraction*, the carbon footprint table, and a *realized abstraction*, as students reflected on the assigned values and considered materiality. Additionally, they examined *the formatting power of mathematics* by analyzing relationships between the table's values, such as how durability affected the carbon footprint; for instance, they noted that a scooter can be used for an extended period.

The teacher asks:

"Does the table feel fair?"

Student 4: That little LOL doll cannot be much more than LEGO, can it?

Student 5: And then the scooter… Scooters are big…

Student 6: But you can have a scooter for a really long time… And there are a lot of other things than just plastic.

Figure 6 Data segment coded from field notes and enacted AWPs for thinking abstraction, realized abstraction, and the formatting power of mathematics

Step 2: Conceptualizing features of AWPs

Step 2 involved conceptualizing features of AWPs, a process that was not linear. It was not solely based on reviewing the coded data segments and their descriptions from Step 1, described above. Indeed, through the initial analytical processes conducted for papers 4, 5, and 6, I gradually developed an understanding of possible features of AWPs. These were initial understandings, meaning that the processes of identifying and conceptualizing features of AWPs did not end with the initial analysis carried out in papers 4, 5, and 6. Rather, this process, as described in the sections above, continued with looking through and across the coded data segments and their descriptions to identify patterns or alignments with existing features, enabling a more fine-grained understanding of some features while replacing others with different ones, and also to identify some features that had not been addressed in papers 4, 5, and 6. Entering the process of a more in-depth analysis of identifying and conceptualizing features of AWPs, I had the following features as a starting point from Paper 6:

- AWPs involve powerful algebraic ideas sociologically speaking
- AWPs mobilize a commitment to sustainability in their intellectual, ethical, and practical aspects
- AWPs have inbuilt dilemmas and value conflicts
- AWPs involve schematic, psychological, and sociological tools as materializers
- Teachers' actions promote students' engagement, exploration, and awareness

A final consideration of AWPs took place, using the data segment descriptions from Step 1 in Stage 2 and revisiting the features of AWPs identified and addressed in Paper 6. This final rethinking allowed me to characterize AWPs in more depth, leading a) to confirm and refine earlier identified features (see Paper 6), and b) to replace features by newly identified features. I illustrate the process that led toward a) and b) below.

Example 1: Evidence for and refinement of a feature
Paper 6 identified the feature *AWPs involving schematic, psychological, and sociological tools as materializers*. Looking across the coded data from the previous step helped me both to find confirmation of this feature and to develop a more fine-grained understanding of it, particularly regarding the meaning of a sociological tool.

More specifically, following Step 1 in Stage 2, I reviewed all segments assigned the *learning model* code. Reading across these segments' descriptions (Step 1 in Stage 2) provided evidence for the feature's validity. When doing this, one piece of evidence of why this feature was confirmed came from reading the descriptions of the data segment (Figure 7), which indicated that using C-rods to represent object values in the carbon footprint table can serve as a materializing tool, a learning model. In particular, the description highlighted how students implicitly drew on the C-rods in their argumentation, reasoning, and exploration of objects underlying sostructures and values. The description also covered that, because the C-rods did not carry fixed numerical values, they seemed to invite students to implicitly use the C-rods, “That little LOL doll can’t be much more than LEGO, can it?”, in their exploration of relationships between objects’ given values and, at the same time, enabled them to propose possible underlying structures of these values. In other words, it was described how the students used the C-rods to reflect on and discuss possible underlying structures of object values, by relating the different objects given values to one another, “But you have a scooter for a really long time...”, and trying to identify a general relationship that seemed to hold across all objects’ given values. As they did so, the tensions between economic and socio-ecological values also became visible. The tension lay in the fact that the economic values, that is, what students were most familiar with, did not hold across all objects’ assigned values. Overall, the C-rods were described as tools that appeared to support conversation at a more general level by avoiding a predetermined underlying structure for the objects’ values, while still materializing students’ exploration, reflection, and argumentation.

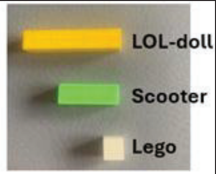
Student 4: That little LOL doll can't be much more than LEGO, can it? Student 5: And then the scooter... Scooters are big... Student 6: But you have a scooter for a really long time... And there are a lot of other things than just plastic.	

Figure 7 C-rods as learning models

At the same time, the segments were additionally coded as *the formatting power of mathematics*, *thinking abstractions*, and *realized abstractions,* which helped to come to a more refined understanding of this feature, particularly in relation to the meaning of a sociological tool. The segments' description based on how these multiple codings helped to a deeper understanding of the meaning of a sociological tool, because in the descriptions, the data became connected, in concrete ways, with the meaning of the codes. The C-rods were described as sociological tools, as students implicitly used them to engage with the salient value conflict between the object's economic and environmental values. This was addressed as a reasoning highlighting the formatting power of mathematics in action. Another part of the descriptions addressed how the C-rods facilitated students' move from a *thinking abstraction* (the carbon footprint table*)* to a *realized abstraction* as they began to explore the underlying abstract structures of the carbon footprint table. In this account, the C-rods were described as a sociological tool because they apparently supported reflection on contextual values (e.g., durability).

Example 2: Identification of a new feature

One of the newly identified features was *AWPs involve different dimensions of abstraction*. Following Step 1 in Stage 2, I reviewed all segments that were assigned all three codes: *thinking abstraction*, *realized abstraction*, and *the formatting power of mathematics*. I did this because I observed a pattern in which the combination of these three codes was repeated across several data segments.

When reading the descriptions, I could see how the three codes were interconnected and, together, seemed to support the simultaneous integration of early algebraic and socio-ecological aspects. The segments'descriptions made in noted that the carbon footprint table (see Figure 6) was a *thinking abstraction*, serving as a model that provided a specific representation of item values (i.e., their carbon footprints). Consequently, the underlying structures that constituted these carbon footprint values were not articulated and were therefore considered hidden. The socio-ecological connection was evident in the carbon footprint values. This *thinking abstraction* was then described as a ground for a *realized abstraction* when students began operating on the given values in the table, exploring possible underlying structures of the carbon footprint values. This was exemplified by students discussing the materiality of the toys/things, and how materiality could be an underlying structure that held across all items in the table (see Figure 7).

The descriptions also captured how the students were trying to establish connections between the objects' values in the table and possible underlying structures that held across all objects, thereby visualizing *the formatting power of mathematics*. It was the interplay between the two forms of abstraction that was recognized as a way of becoming aware of the formatting power of mathematics, namely, the possible hidden structures of the table.

The connection between the codes was that, in response to the teacher's question, students engaged in exploration, interpretation, and reasoning to determine which object characteristics influenced the values assigned in the carbon footprint table. This process involved a *thinking abstraction*, the carbon footprint table, and a *realized abstraction*, as students reflected on the assigned values, considering materiality. Additionally, they examined *the formatting power of mathematics* by analyzing relationships among the table's values, such as how durability affected the carbon footprint; for instance, they noted that a scooter can be used for an extended period. Against the background that these descriptions capture how two dimensions together mobilized students' reflections on the formatting power of mathematics, the feature that AWPs involve different dimensions of abstraction was identified as a new feature.

Ethical considerations

The ethical considerations in this chapter pertain to the second phase of my doctoral research. Ethical issues relevant to my licentiate thesis have already been addressed in Fred (2019).

The decisions underpinning my doctoral studies were guided by the Swedish Research Council's (Vetenskapsrådet, 2017) ethical principles for social science research. All participants, namely the in-service and pre-service teachers engaged in the collaborative research process, as well as the students involved in AWPs tryouts and their guardians, provided informed, written consent. Prior to giving consent, participants received comprehensive information regarding: a) the overarching aims of the research project, b) the conditions and expectations associated with their participation, c) their right to withdraw from the study at any time without obligation to explain, d) the intention to publish the research findings in scholarly publications and present them at relevant professional forums, and e) the assurance that all participants would remain anonymous in published materials.

According to the Swedish Ethical Review Act (2003:460), the primary aim of an ethical review is to safeguard individuals and uphold human dignity throughout the research process. Pondering these points led to the conclusion that no ethical review was required. In the inquiry group, we discussed how teachers should respond if students, during a discussion, express political or other types of viewpoints (their own or their parents'). We agreed that such situations should be handled carefully, with the teacher guiding the conversation back to the topic and maintaining a respectful, balanced discussion. The teachers who participated in the inquiry group noted that students at their school were accustomed to discussing issues of democratic citizenship and were familiar with appropriate ways to engage in these conversations.

Andrée et al. (2020) argue that ethical considerations in educational research extend beyond the legal regulations of ethical research conduct mentioned above. Therefore, ethical reflection in this case includes both consideration of the content and of students in relation to AWPs, as well as research practice collaborations when imagining AWPs.

Ethical reflection concerning content and students

In this section, I address the ethical dimensions of students' work with AWPs, including both those participating in the empirical trials conducted for this thesis and those who may encounter AWPs in future educational contexts.

In the empirical study, the research team invented, designed, planned, and reflected on a new type of problem for early algebraic activity—AWPs—to invite students to engage in algebraic thinking more broadly. Through this design, students may be encouraged to identify, consider, and discuss ethical issues as participants in society when approaching the world through mathematics (see Ernest, 2019), especially in relation to socio-ecological concerns embedded in their life-worlds. This introduces ethical challenges.

One ethical challenge involves navigating the tension between addressing socio-ecological issues and avoiding unintended harms—such as feelings of guilt or exclusion—when students perceive that their own or their families' views diverge from perspectives treated as normative in the classroom. Research by Léger-Goodes et al. (2022) shows that exposing children to socio-ecological problems can evoke fear and anxiety. Yet, as Bergdahl and Langmann (2022) argue, such concerns must still be addressed in teaching. Students require a safe space in which they can explore and process worries collectively—a space where the goal is not solely to seek solutions but also to reflect, clarify uncertainties, and engage with conflicting values.

When students are working with AWPs, the above-described tension can arise. The issue at stake is the tension between the need to address these kinds of topics and the need to avoid fear and anxiety. Therefore, the goal was to create a safe space for students to engage with these topics. To that end, the inquiry group, during their meetings, discussed in detail how to address socio-ecological issues in the classroom to enable students to confidently express their values, feelings, and conflicts of interest. Several aspects of creating a safe environment were highlighted, including the fact that the students who participated in the tryout were already accustomed to discussing this type of issue. A stated goal at the school was to develop active democratic citizenship, which is also central to socio-ecological issues.

Another aspect was that the teachers, together with the students, regularly watched *Lilla Aktuellt*, a TV news program for children, which is supplemented with educational material to discuss, for example, societal issues and understanding of the news. The teachers had experience in handling these kinds of issues and the tensions that could arise. Finally, a socio-ecological theme was chosen that was within the framework of the students' life-worlds and that they had previously worked with in other subjects.

Another approach discussed was the use of fictional cases, allowing students to analyze the consequences of these fictional students' choices, for example, whether they choose to buy new, secondhand, or refurbished items (see papers 4 and 6). This approach was seen as shifting the focus toward understanding trade-offs and evaluating courses of action without assigning blame or causing discomfort.

Ethical reflection concerning research and practice collaborations

This section addresses ethical dimensions of research practice collaborations, with particular attention to relationships between actors, power dynamics, shared research objects, and methodological values.

Ethical challenges arise when research involves people, and as relationships between actors may evolve throughout the process (Andrée et al., 2020), researchers need to consider what these challenges imply in practice. At a general level, such challenges concern a "commitment to hold oneself responsible for how one tries to influence other people's learning" and an acknowledgement of being part of a world connected with others in an endeavor to undertake enquiries with them (Andrée et al., 2020, p. 136). In this study, the aim of empowering students as future citizens is pursued alongside the aim of opening possibilities for curricula and pedagogies that promote early algebraic thinking. It therefore follows that the research process is intended to influence participants' (in-service teachers, pre-service teachers, teacher educators, and researchers) understandings of what early-grade algebra teaching could be. Across the project phases, this raised issues of hierarchy and dependency among participants. When values such as status or professional titles are at stake, participants may position themselves differently within the collaboration, shaping how they influence one another at both conscious and unconscious levels (Andrée et al., 2020). In this project, tensions around dependency and hierarchy were observed between the researcher and the teacher educator (me), between the researcher and the in-service teachers, between pre-service teachers and the teacher educator, between in-service and pre-service teachers, and between teachers and students. I approached the situation by positioning myself as one of the participants, rather than as the most knowledgeable individual. During inquiry group meetings, I consistently highlighted the fact that each member brings unique experiences and expertise, which together form a solid foundation for envisioning AWPs.

The above also relates to how tensions linked to "a lack of credibility between researchers and practitioners" (Rovio-Johansson, 2020, p. 1) became visible during the formation of the research team. These tensions concerned both how participants perceived one another and how they understood their own positions and contributions. They also reflected "historical asymmetric power relations" (Lillejorda & Børtea, 2016, p. 558), shaping who was regarded—by others and by themselves—as most knowledgeable and therefore most powerful. Responding to these dynamics required deliberate efforts to balance power and responsibility and to negotiate when initiating and sustaining collaborations (Hamza et al., 2018). Over time, these dynamics shifted within the inquiry group and became less visible. This underscores the need to address such tensions as an ongoing part of collaborative work and to be aware of how conditions for participation are established and sustained.

The collaboration can also be analyzed in terms of work on a shared research object. Magnusson and Malmström (2022), emphasize "the importance of having a mutual research object, and researchers' and teachers' collaborating in planning and developing the teaching material" (p. 5). Such an object does not emerge automatically; it is constituted through collaborative work in which implicit and explicit issues are identified and addressed. In the research team for my PhD project, this became evident over approximately one year as AWPs were jointly imagined and iterated. Conceptualizing something required participants to step beyond their established fields of knowledge and "jointly create new knowledge" (Lillejorda & Børtea, 2015, p. 558). In relation to the research team's work, it became clear how difficult it is to step beyond established fields of knowledge. And it became clear that the research team made progress in the imagination of AWPs when it was able to do so.

Further methodological values in participatory work concern relations of dependency among researchers, in-service teachers, pre-service teachers, students, and teacher educators. These include decisions about what counts as data (and what does not), how data are analyzed and interpreted, and how to reconcile potentially competing values—such as ownership of research outcomes and conventional ethical requirements regarding integrity and anonymity (Andrée et al., 2020). In this project, participation positions and the dependencies and loyalties they entail were also at stake. As a researcher, I participated in the collaborative work of imagining AWPs while also analyzing that work, which raises questions about how to negotiate values tied to the research community (e.g., what kinds of results are valued) alongside values tied to participants (e.g., what participation offers pre-service and in-service teachers) (Andrée et al., 2020). More broadly, this concerns recognizing all research-team participants as agents rather than objects of study (Newton & Burgess, 2008; Carlgren, 2012). Accordingly, the intention was not to categorize participants as "persons who think" or "persons who do," but to treat all participants as contributors exploring possibilities in the design of curricula and pedagogies that promote algebraic thinking.

Paper 4: Algebraic wicked problems: Bridging the gap between early school algebra and socially relevant issues

Fred, J., Valero, P., &, Van Steenbrugge, H. (2023). Algebraic wicked problem: Bridging the gap between early school algebra and socially relevant issues? In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, & E. Kónya (Eds.), *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)*. Alfréd Rényi Institute of Mathematics & Eötvös Loránd University of Budapest.

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The findings show that contextualization comprises layers that go beyond simply providing students with a mathematical model as context. The findings also identify built-in dilemmas, which were seen as central to enabling the integration of algebraic thinking and socially relevant issues to emerge in students' work with AWP. The paper also recognizes the importance of these dilemmas, prompting a closer analysis of their nature and function.

IV

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Algebraic wicked problems: Bridging the gap between early school algebra and socially relevant issues?

Jenny Fred, Paola Valero, Hendrik Van Steenbrugge

Algebraic wicked problems: Bridging the gap between early school algebra and socially relevant issues?

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This paper aims to discuss an advance in the direction of bridging the gap between early school algebra and socially relevant issues using the idea of wicked problems. We do so by focusing on the possibilities and challenges of working with the idea of algebraic wicked in the context of planning, teaching, and reflecting on algebra lessons for young children of 7-9 years old. This endeavor is a collaboration between the researcher/teacher educator, teacher students and in-service teachers following the critical research methodology of imagining and trying to create something that does not yet exist. This paper represents partly how far we have come to formulate an algebraic wicked problem that can invite students to engage into a cultural and sociological algebraic activity and as such reports on what we have learned from trying to design and enact algebraic wicked problems.

Keywords: Early algebra, algebraic thinking, critical mathematics education, socio-political perspective, wicked problems.

Introduction

In our contribution to CERME 12 (Fred et al., 2022), we have pointed out that most of the existing research literature on early algebraic thinking mainly adheres to a view of algebra as powerful in logical and psychological terms; and seldom as culturally and sociologically powerful mathematical ideas. Drawing on Skovsmose and Valero (2008), early algebraic thinking is understood in line with an interpretation of algebraic ideas as culturally and sociologically powerful when early algebraic thinking is conceived as a resource or a tool for concrete critical thinking and action, and as an analytical tool for exploring models of "reality" and for creating an awareness of how those models only visualise certain "things" and leave other things unnoticed (Cai & Knuth, 2011; Kieran, 2018). Young students should thus explore, interpret and reason about functional relationships between quantities in given models of students' "reality" (compare Blanton et al., 2015; Coles & Ahn, 2022).

We have also discussed earlier (Fred et al., 2022) the idea of connecting early algebraic thinking with *wicked problems* (Rittel & Webber, 1973; Steffensen, 2021) as a possibility to invite pupils to connect early algebra with culturally and sociological forms of citizen engagement. Wicked problems are problems that involve controversies that trigger arguments instead of simple solutions and are widely used in sustainability research (Steffensen, 2021). This paper has the aim of discussing an advancement in this direction by focusing on the possibilities and challenges of working with the idea of *algebraic wicked problems* (AWP) for the practice of teaching early algebra for young children (age 7-9) with a focus on environmental issues. The following question will be discussed in this paper: What can we learn from trying to design and enact algebraic wicked problems for and with young school children (age 7-9)?

Algebraic wicked problems

The term wicked problem was introduced by Rittel and Webber (1973) to problematise what they called 'tame problems' in the natural sciences, in contrast to complex societal-technological-scientific problems. Along this line, algebraic 'tame problems' can be understood as many of the problem situations used in typical school mathematics word problems for pupils to do some algebra, such as, for example, solving an equation without context. In a tame problem it is clear what the pupils should do, and it is also clear "whether or not the problems have been solved" (p. 160). Wicked problems are problems which can never be definitely solved or that at most are re-solved repeatedly. In other words, in working with these kinds of problems it is neither clear for the pupils what to act upon or what a solution might be, and there are also personal and environmental aspects at stake besides the mathematics. Thus, working on such problems involves both mathematical and societal framings. Pupils could be encouraged to negotiate disagreements as well as being open to different ways of approaching and framing the problem. That is, wicked problems are fraught with dilemmas that need to be addressed as part of facing the problematic situations.

We are tapping into an underexplored area in research and educational practice, which Skovsmose and Borba (2004, p. 211) would describe as researching "what is not the case —but what could be a possibility". Thus, we take Rittel and Webber's (1973) idea of wicked problem as a rhetorical, creative, and critical/emancipatory tool (e.g., Lönngren et al., 2021) to create early algebraic practices that invite pupils to consider and act on important problems and concerns as members of a society in a critical way (e.g., Skovsmose, 1994). In the following section, we describe how we study what is not yet the case.

Methods

Skovsmose and Borba (2004) propose a model to develop critical, collaborative, pedagogical practice while researching non-existing, yet-to-come situations. A *current situation (CS)* can be transformed into an *arranged situation* (*AS)* with inspiration in a theory-grounded, ideal *imagined situation*. The collaboration of teachers and researchers in moving between these situations can be characterised by qualities such as *pedagogical imagination* (*PI*), *practical organisation* (*PO*) and *explorative reasoning* (*ER)* (see figure 1). The model offers methodological tools for thinking, doing, and talking about critical elements that appear in the exploration process when challenging an existing and/or dominant practice, as in this case, exploring the possibilities of how to place algebraic thinking in a larger context and to provide tools to pupils to become active citizens.

In our case, since 2021, such a critical iterative research process has been initiated in a group of four in-service teachers, four teacher students, and one teacher educator/researcher. Together we have tried to create AWPs for pupils in grades 1-3 (aged 7-9) in Sweden. The AWPs have also been enacted in classrooms. Three cycles of iteration have been conducted; we report about the third iteration.

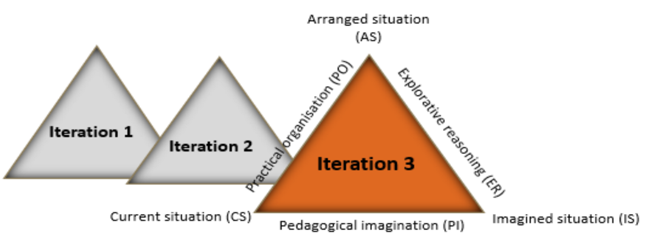

Figure 1: Skovsmose and Borba's (2004) model of Critical mathematics education-research.

In the third iteration the *current situation* consists of both the common interpretation of algebraic teaching in the early grades and what the research group learned in iteration 1 and 2. The *imagined situation* concerns a new imagination, including knowledge from the previous iterations of new possible design of curricula and pedagogies that empower pupils as future citizens and at the same time promote pupils algebraic thinking. The *arranged situation*, finally, concerns the experiment of staging the new possible design, a staged AWP. The pedagogical imagination refers to the relationship between the common interpretation of algebraic teaching in the early ages together with what the research group learned in iteration 1 and 2, the *current situation*, and what kind of AWP we can imagine, the *imagined situation*. The practical organisation refers to the organisation of or staging of the newly arranged AWP. The explorative reasoning refers to the analysis of the *arranged situation* in iteration 3 in the light of the *imagined situation*, in other words, the formulation of AWP based on the observations of the *arranged situation* in which the imagined AWP was staged.

In this paper, the pedagogical imagination and explorative reasoning are working as analytical focuses/lenses with the aim of learning more about how AWPs potentially can bridge the gap between early algebra, and socially relevant issues in terms of environmental issues. The *pedagogical imagination* provides an analytical lens to describe the thinking in the research group when planning to teach an AWP. We drew on several sources of data: a) transcriptions from two group meetings planning for and discussing the idea of AWP, b) observation notes from the *arranged situation* as it was enacted in classrooms, c) pictures (no faces) from pupils working in the *arranged situation*, d) pupils' solutions to the AWP enacted in the *arranged situation,* and e) transcriptions from two group meetings reflecting on the lesson to better shape an AWP. In the analysis, segments of the transcriptions of the group meetings conversations were marked to indicate their particular focus such as algebra, didactics, and critical citizenship. The software MAXQDA was used in the work with the transcripts. In the four analysed group meetings, 37 excerpts were analysed; 16 of these relate to the pedagogical imagination and 21 to the explorative reasoning.

In what follows, the analysis is unfolded to describe how an AWP was constructed and what characterised it as a result of the *pedagogical imagination*, and later how its enactment in the arranged situation was discussed through the *explorative reasoning* in the collaborative team.

Planning of the lesson to create an AWP

In the *pedagogical imagination* of the AWP the group struggled with several issues. One was grasping pupil-oriented contexts, for pupils aged 7-9, in relation to environmental issues that still included early algebra. After some discussions the research group agreed on a context which concerned an environmental issue relevant to younger pupils' lives. Based on the issue, one of the teacher students came up with the idea of giving the pupils a climate budget.

Teacher student 1: Thought a little in the form of giving the pupils a climate budget or maybe 100 SEK and so they need to prioritise... I want a mobile phone, or I want this or that toy but then there should be so many options that you can't choose everything/…/. So, say a pupil wants a mobile phone, for example, then it will be like you can't afford much else. But there will be different amounts of solutions because the pupils have different priorities. So, it will be interesting to compare you think this was important and you thought this was important.

The suggestion was to put the pupils in a problem situation where they were asked to choose between already determined things/toys and how their choices also should be limited by a climate budget. This was then further developed with the idea of giving the pupils a table consisting of things/toys carbon footprints values. The table (see Figure 2) was seen as a possible model of reality, within a pupil-oriented context, while most pupils are used to wishing for things/toys for their birthdays or for Christmas.

You get a carbon footprint budget with the value of two orange Cuisenaire rods	
TV-game	Carbon footprint: 1 blue rod
Cell phone	Carbon footprint: 1 black rod
Loll-doll	Carbon footprint: 1 yellow rod
Widget toy	Carbon footprint: 1 yellow rod
Scooter	Carbon footprint: 1 green rod
Lego	Carbon footprint: 1 white rod

Figure 2: The Carbon footprint budget

The idea of using Cuisenaire rods (C-rods) came up when discussing how algebraic thinking could come into play in the pupils' work. The idea was that the C-rods would function as representations of carbon footprint values. Using C-rods as representations of values was seen as a facilitator to invite the pupils to explore, interpret and reason about underlying structures which affected the toys/things given values.

A second struggle was how to create a dilemma that would invite the pupils to simultaneously integrate algebraic thinking and environmental issues in their work. The research group came up with a task, in which the pupils in pairs should be asked to select toys, from the given table, limited to a carbon footprint budget of two orange C-rods (See upper row in Figure 2). The anticipated dilemma in the task consisted of three features, a) pupils needed to create one joint budget, b) then the pupils had a budget that would limit them in the selection of number of things/toys and c) as a second step, the pupils should be asked to halve their already limited budget. The research group's idea with before mentioned features was to challenge the pupils to argue, prioritise, and negotiate their choices by using algebraic thinking as a tool and then the values were in terms of carbon footprint environmental issues would become integrated in the pupils' work.

Reflecting on the lesson to better shape an AWP

In the *explorative reasoning* the discussions focused on both failures and success of the anticipated dilemmas, how a possible new *arranged situation* would be designed, creating possibilities for integrating algebraic thinking and environmental issues.

One *dilemma* discussed concerned when the pupils were asked the question "Does the table feel fair?". The perception of the teacher teaching and the teacher educator/researcher observing the arranged situation was how the pupils were both exploring possible underlying structures of the LOL doll's, the Lego's, and the Scooter's given values (see figure 2):

Pupil 4: That LOL doll can't be much more than Lego, can it?
Pupil 5: And then the scooter... Scooters are big...
Pupil 6: But you have a scooter for a really long time... And it's a lot of other things than just plastic.

When Pupil 6 expressed that you can have a scooter for really long time this was recognised by the research group as an awareness of aspects of sustainability in terms of different materials as well as an explanation of why a scooter has a lower value then the LOL doll.

In another episode a pupil asked if a white C-rod represented the value of one Lego brick or, a big or small Lego set. The research group recognised this as the pupil was exploring the relationship between the Lego's "size" and its carbon footprint. An underlying concern, according to the research group, could have been that if one white C-rod was representing only one Lego brick the carbon footprint for the Lego would be high in relation to the other things'/toys' carbon footprint.

The *second dilemma* the research group discussed was when pupils were asked to create a joint budget in pairs. This was however, by the research group, seen as a failure, because the pupils ended up only exploring the quantitative relationships between the C-rods rather than exploring which underlying structure (in terms of environmental aspects) affected the toys'/things' values, as was the intention.

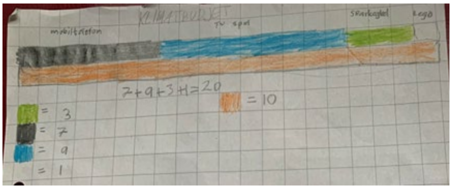

Figure 3: Pupils' solution

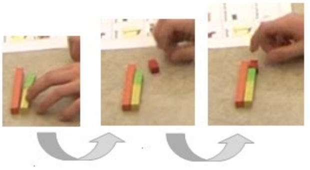

Figure 4: Pupil creating equality

Another reason was how the pupil (in figure 4) was testing if adding a red C-rod would create an equality and when it did the pupil stopped their exploration. This phenomenon was also observed in other pupil's work:

Teaching teacher: Then it was a bit like that because they couldn't choose two video games, didn't work out, okay then, then I'll take a mobile instead and then they filled up with it.

This was also recognised as the pupils only were exploring the quantitative relationships between the C-rods rather than exploring which underlying structure (in terms of environmental aspects) affected the toys'/things' values. The research group made the conclusion that this planned dilemma became what Rittel and Webber (1973) would describe as a 'tame problem', an ordinary math task where the pupils just said "OK, we're going to create an equality" by finding C-rods that together are as long as one orange C-rod.

The discussions also concerned *how a possible new arranged situation would be designed.* One possibility raised was how there could be an example of an unknown pupil who wants too many things/toys, and then the task for the pupils would become to help the pupil to rethink because he/she cannot consume as much as he/she wants. Another possibility was having three different already made carbon footprint budgets represented by C-rods, made by fictional pupils, and where the pupils would be invited to reason about the already made budgets. In relation to this it was suggested that, for example, the value of a mobile phone should be represented with different C-rods depending on if it was a new or used/refurbished one. In other words, the mobile phones carbon footprint values needed to be seen in relation to if it was a new or used/refurbished one. This was seen as a possibility to invite the pupils to explore, interpret and reason about how the use of Planet Earth's resources as a corresponding aspect in the different the budgets, then a new mobile phone will have a higher carbon footprint then manufacturing a new mobile phone take more of our planet's resources. This arrangement was seen as an imaginable way of inviting the pupils in discussions concerning models' potential of visualising only certain "things", leaving other "things" unnoticed, where the unnoticed things in this case will be if the mobile phone is a new or used/refurbished one and how this plays a role in relation to carbon footprint impact.

Discussion

The aim of this paper is to discuss what we can learn from trying to design and enact an algebraic wicked problem (AWP). In this discussion we will highlight the contextualisation and the built-in dilemmas in relation to elements of algebraic thinking coming into play or not coming into play in pupils' work with AWP.

In Fred et al. (2022) the contextualisation was highlighted as crucial when enlarging the idea of algebraic practices to involve also socially relevant issues. The contextualisation phase, however, consists of different layers which goes deeper than just giving the pupils a carbon footprint table. In our case, not only the choice of data seemed to be important but also how the value of a toy/thing was not of what the pupils are most familiar with, a financial value. This we understand as a built-in dilemma then it seemed to invite the pupils to explore, interpret and reason about other possible corresponding relationships between the different toys'/things' values. However, it was the teacher's question "Does the table feel fair?" who invited Pupil 6 to reason about aspects of sustainability in terms of different materials and how this affected the value of a Scooter and a LOL-doll. Another example was when the pupils were exploring the relationship between the Lego's "size" and its carbon footprint value. In the Lego-case the pupils' exploration could be understood as an indication that they were trying to figure out the corresponding values between Lego and the other things/toys by exploring the value in relation to size/quantity. We consider both examples of pupils reasoning as an indication that they were "thinking at the functional-particular level conceptualised a functional relationship as a set of particular relationships between specific corresponding values" (Blanton et al., 2015, p. 530). This then they were exploring, interpreting, and reasoning about which corresponding values affected the scooter's, the LOL doll's and the Lego's given values in the carbon footprint table. In the first case were pupil 6 was using the sustainability of different material to reason about the relationship between the things/toys given values which we interpret as an example of when algebraic thinking and environmental issue becomes integrated simultaneously in pupil's work. In the second case, to make the pupils also to integrate environmental aspects the teacher would have needed to challenge the pupils with a question or a statement that directed their awareness also to these aspects. Altogether, from the perspective of our try-out the different layers of contextualisation, the design of built-in dilemmas together with teacher's questions or statements appeared to play a crucial role for algebraic thinking and the environmental issues becoming integrated simultaneously in the pupils' work.

The result also shows how easy it is to fall into the trap of what Rittel and Webber (1973) would describe as a 'tame problem', an ordinary math task. When the pupils were asked to create a joint budget there where nothing that invited them to explore underlying structure of what affected the toys'/things' values. So, it is no surprise that only algebraic aspects came into play in the pupils' work. This bring us back to complexity of AWP and the different layers that need to be taken into consideration when designing, staging as well as maintaining pupils work with the AWP. In the groups reflection on the lesson to better shape an AWP they were imagining new possibilities of how to integrate algebraic thinking and environmental issues simultaneously in the pupils' work with carbon footprint budgets. One imagination was to give the pupils three different budgets, already made by fictional pupils, and where the mobile phone would have different values depending on if it was a new or used/refurbished one. However, if we want algebraic thinking and environmental aspects to become integrated simultaneously in the pupils' work this is not enough. We also need to imagine questions or statements the teacher could ask/express to challenge the pupils to pay attention of both in their work. It does not happen by itself!

Finally, it can be said that the possibilities of generating and successfully bringing AWPs in classrooms has challenged the different types of people involved. This effort has required not only to go beyond usual imaginations of situations that are used to invite into algebraic thinking, but contexts of significance in cultural and sociological terms. It also requires that the contexts and the way teachers work with them pose dilemmas where the characteristics of algebraic thinking are mobilised. So far, this path has shown that collective work may show new ways to connect early school algebra with problems and situations of relevance for children as future citizens to make it culturally and sociologically powerful.

Acknowledgment

This paper reports on research conducted within the graduate school *Relevancing mathematics and science education* (RelMaS), funded by the Swedish National Research Council (VR3100-2908).

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V

Paper 5: Connecting early algebra and sustainability through dilemmas in Algebraic Wicked Problems

Fred, J., Van Steenbrugge, H., & Valero, P. (2024a). Connecting early algebra and sustainability through dilemmas in Algebraic Wicked Problems. In Twohill, A., Isler-Baykal, I., Knuth, E., Ribeiro, A. J., & Miller, J. *Proceedings of the 15th International Congress on Mathematics Education (ICME15), Topic Study Group 1.2: Teaching and learning of early algebra* (pp. 16–23). Sydney, Australia.

This paper discusses dilemmas and their capacity to foster early algebraic thinking and sustainability in students' engagement with AWP, as well as to inform the future development of AWP.

The following question was addressed: Dilemmas and their potential to simultaneously connect early algebraic thinking and sustainability in pupils' work with AWP.

The findings show the multidimensional nature of dilemmas. They also indicate that dilemmas are built-in, incompatible conditions marked by conflicts between opposing values. Such value conflicts encourage students to examine, analyze, and reflect on the relationships among quantitative data within a model of reality.

The 15th International Congress on Mathematical Education
Sydney, 7-14 July, 2024

CONNECTING EARLY ALGEBRA AND SUSTAINABILITY THROUGH DILEMMAS IN ALGEBRAIC WICKED PROBLEMS

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Early algebraic thinking can be connected in significant ways to sustainability issues through algebraic wicked problems (AWP). AWP are problems that mobilize children to think algebraically while experiencing dilemmas on values at stake in sustainability. Both algebraic thinking and sustainability issues become inseparably connected. From an ongoing analysis of 10 research group meetings over a school year in a group consisting of teachers, pre-service teachers, and researchers, and based on observed lessons, we explore such dilemmas as a central characteristic of AWP. This provides details into what may be important aspects in the further design of AWPs.

INTRODUCTION

As society gets more complex and even young children are confronted with the challenges of sustainability and climate change, it becomes necessary to make explicit connections between mathematics teaching and learning, such as the development of early algebraic thinking, and the ways in which such early algebra plays a role in preparing children to address current societal and environmental difficult situations. Barwell et al. (2022) highlight how the current sustainability crises — in environmental, social and economic terms — do not get enough attention in mathematics education. Solares-Rojas et al. (2022, p. 204) argue that young generations need to "build tools, options, and paths to prepare /…/ to confront" these issues, and that educational strategies need to be developed to prepare pupils to face these issues. In this line of thinking, the concern of mathematics education for "the socio-ecological" (e.g., Coles, 2023) cannot be limited to the use of data concerning sustainability, climate change or real-world 'environmental impact' as a context in the teaching of mathematics (e.g., Boylan & Coles, 2018). Pupils also need to be challenged to go "behind the data", and in reflective dialogue discuss possible underlying structures/values of the data (Barwell, 2013). The current predicaments of sustainability invite us to push the boundaries of the significance of mathematics education for new forms of critical citizenship.

In our current research work, we particularize these challenges to early algebra. Our previous review of early-algebra research pointed to the dominance of studies that justify the need to develop early algebraic thinking in terms of their contribution to the logical and cognitive development of pupils (Fred et al., 2022). It also showed a lack of studies about early algebra in connection to cultural and societal issues such as sustainability. In an attempt to tackle such shortcoming, we have explored the possibilities of working toward early algebra teaching and learning that takes the challenges of sustainability as previously described (Fred et al., 2023a).

For this purpose, Jenny Fred built a research team consisting of four grade 1-3 teachers, four pre-service teachers, and three researchers (the three authors of this paper). We collaborated (Fred et al., 2023b) on the process of inventing, designing, planning, and reflecting on a new type of problem for early algebraic activity: Algebraic Wicked Problems (AWPs). The collaboration around the AWPs took place during one school year, during which 10 research group meetings were closely documented and studied.

In our explorative work of creating both the notion of AWP and formulating concrete examples, we drew on Rittel and Webber's (1973) idea of a *wicked problem* as a rhetorical, creative, and critical/emancipatory tool (e.g., Lönngren & van Poeck, 2021). Already in the early 1970s, the notion of wicked problem was used to refer to complex, multidimensional predicaments that are not easily solvable by simple policy intervention; indeed, wicked problems may not even be "solvable" at all (Rittel & Webber, 1973). The term was introduced to problematize what they called 'tame problems' in the natural sciences, in contrast to complex societal-technological-scientific problems, one of which can be the multilayered, multilevel problem of sustainability and climate change in current times. Building on this idea, science educators for some time (e.g., Lundegård & Caiman, 2019) — and mathematics educators more recently (e.g., Steffensen et al., 2021) — have used the notion of wicked problems to create situations of teaching and learning that go beyond many of the 'tame problems' used in typical school word-problems. The usual problems that pupils meet when working with algebra, such as solving an equation without context or figuring out the general rule for a number pattern, can be characterized as tame problems. In a tame problem it is clear what the pupils should do, and it is also clear "whether or not the problems have been solved" (Rittel & Webber, 1973 p. 160). In contrast, wicked problems can never be definitely solved or, at most, are re-solved repeatedly because they involve multiple dimensions in connection, values mobilized, and viewpoints at stake.

The explorative work in the research team consisted of taking on board the notion of wicked problem to reinvent algrebraic problems for early school algebra. Furthermore, if the challenges of sustainability were not to be taken as a simple context of formulating tame school algebra problems, but as a possibility to connect algebraic thinking as a significant tool to explore the wickedness of sustainability, then pupils should be invited to integrate algebraic thinking and sustainability in their work with the AWPs. Algebraic thinking should work as a resource or a tool for concrete critical thinking and action (Skovsmose & Valero, 2008), as an analytical tool for exploring models of "reality", and for creating an awareness of how those models only visualize certain aspects of a reality and leave other unnoticed (Cai & Knuth, 2011; Kieran, 2018). That is, the pupils should be invited to explore, interpret, and reason about functional relationships between quantities in given models of pupils' "reality" (e.g., Coles & Ahn, 2022).

How to make algebraic thinking and sustainability issues becoming simultaneously integrated in pupils' reasoning is not an easy matter. In our try-outs of different formulations of AWPs with pupils, although the framing of a AWP consisted of these two aspects, the pupils did not necessarily connect them with one another. In our further work on that challenge, we have experienced how *dilemmas* can open a possibility to make the merging of algebra and sustainability happen. Dilemmas are built-in incompatible condition in a problem which requires pupils to analyze and reflect on quantitative data in a model of reality (Davydov, 2008). This paper explores dilemmas and their potentiality for connecting early algebraic thinking and sustainability simultaneously in pupils' work with AWP, and for further designing AWPs.

METHODOLOGY

We drew on multiple sources of data that all are related to the research team's invention, design, planning, and reflection on AWPs.10 research team meetings that consisted of planning and reflecting on working with AWPs were recorded and transcribed; field notes of lesson observations (here we particularly report on grade 3 lesson for about 20 pupils). Transcribed research team meetings were marked to indicate whether the particular focus of the conversations were involving: aspects of algebra and algebraic thinking, aspects of teaching and learning, aspects of sustainability, or aspects of the purpose of education for critical citizenship. The data as a whole was then analyzed to understand the instances where the aspects of algebraic thinking and sustainability overlapped, and how they became integrated in the pupils' work with AWPs as well as in the discussion in the research team. For this paper we have chosen to concentrate on two concrete examples to tease out how dilemmas emerged and turned out to be an important element in thinking what may characterize AWPs.

In what follows, we introduce a situation of work with an AWP that was constructed in the research team. Then, we provide two descriptions of lesson fragments where pupils work on AWPs. One of the descriptions exemplifies how a dilemma emerges but does not connect algebraic thinking and sustainability; while the second description shows the opposite. The descriptions are structured to highlight: 1) the built-in incompatible condition emerging, 2) the setting in which it emerged, and 3) how algebraic thinking and sustainability related in the situation. Finally, we discuss some central dimensions of dilemmas that we have identified so far. Our data suggests that these dimensions may be helpful in further designing AWP where algebraic thinking and sustainability intertwine.

EMERGING DILEMMAS

The teacher starts the introduction of the situation by asking the pupils if they know what carbon footprint is. One pupil answers that it relates to environmental damage as for example waste, carbon dioxide and exhaust fumes. The pupil continues this explanation by stating that this kind of environmental damage is what causes the Antarctica to melt which, in the future, will make the the worlds water level to rise. In this introduction, the pupils are also invited to a short discussion concerning what a budget is and how you cannot "buy" more than your budget. Finally, the pupils are asked to create a joint carbon footprint budget, in pairs, with things/toys they want for Christmas. The pupils are told that they can only choose things/toys from a given carboon footprint table.

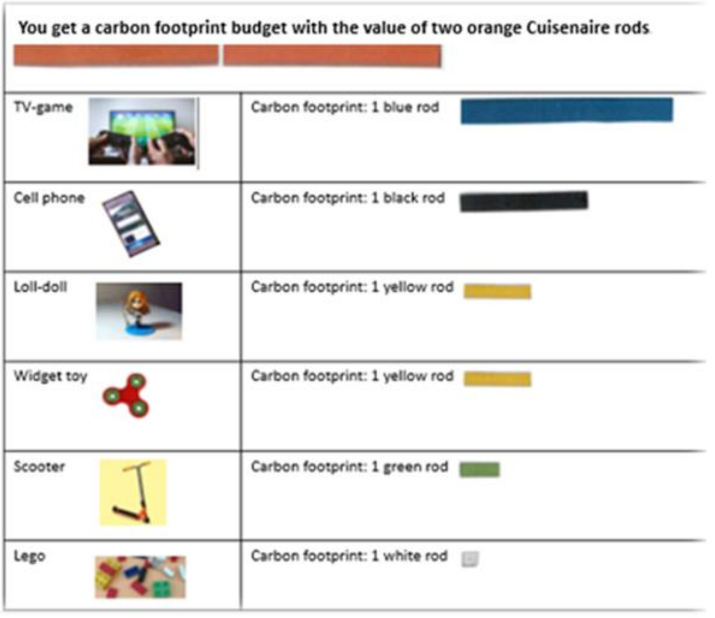
Example 1: How much Lego?

The *built-in incompatible condition* in this example consists of a *value conflict* between Lego's economic value and its carbon footprint value: Lego is expensive to buy but has a low carbon footprint value.

In *setting the scene* the teacher introduces the idea of a carbon footprint budget that pupils have at their disposal. It consists of two orange Cuisenaire rods (C-rods) that the teacher shows to the class (Figure 1). The teacher continues presenting the pupils with a carbon footprint table of different things/toys (Figure 2), and explains that the things/toys in the table have been given different values in terms of their carbon footprint, represented by different C-rods. When explaining the table, the teacher shows the C-rod that represents the Lego's carbon footprint value by showing <u>one</u> white C-rod to the class (Figure 3).

Figure 1: The teacher showing the budget (reconstructed picture)

Figure 3: The teacher showing the Lego's carbon footprint (reconstructed picture)



You get a carbon footprint budget with the value of two orange Cuisenaire rods

TV-game	Carbon footprint: 1 blue rod
Cell phone	Carbon footprint: 1 black rod
Loll-doll	Carbon footprint: 1 yellow rod
Widget toy	Carbon footprint: 1 yellow rod
Scooter	Carbon footprint: 1 green rod
Lego	Carbon footprint: 1 white rod

Figure 2: The carbon footprint table

A *value conflict* emerges around the question if the C-rod represents the value of one Lego brick or a whole set of Lego bricks.

Pupil 1: What is one [emphasizes is and one] Lego?

Teacher: This is probably a box of Lego. Not just a piece but... What do you say? What do you think?

Pupil 2: One Lego bag?

Teacher: One Lego bag? And...

Pupil 1: I think it is one Lego set.

Teacher: Yes, one Lego set. But not the biggest.

The exploration, interpreting and reasoning above concerns primarily the relationship of correspondance between the given value of one C-rod and the amount the Lego (one brick or a package of bricks). The discussion does not connect to any sustainability aspect. Instead, it seems that the potential sustainability issues that could be connected to the amount of Lego remain working as part of an unreflected context. The focus here is on quantities and their corresponcence in value.

Example 2: How come the cost of lol-dolls, Lego and scooters?

The built-in incompatible condition in this second example is that, in the table (Figure 2) the value of a toy/thing is expressed in the currency of carbon footprint while pupils are familiar with currencies for economic value. Also, the value in carbon footprint terms does not seem to align with the toy/thing' economic value. That is, an "expensive" toy/thing in money is "cheap" in carbon footprint.

In setting the scene, the teacher asks the question: "Does the table feel fair?" and simultaneously points at the carbon footprint table (Figure 2), that is displayed on the whiteboard in front of the pupils. A *value conflict* emerges when the pupils start, as a response to the teacher's question, to explore, interpret, and reason about which characteristics of the objects are affecting the toys/things given values in the carbon footprint table.

Pupil 4**:** That little lol-doll can't be much more than Lego, can it?

Pupil 5: And then the scooter... Scooters are big...

Pupil 6**:** But you have a scooter for a really long time... And there are a lot of other things than just plastic.

Pupil 6's expression "you can have a scooter for really long time", can be recognized as an awareness of durability of a toy that is important for sustainability. As part of the series of comments, it also provides an explanation for the corresponding relationship between the scooter's and the lol doll's carbon footprint value. We interpret such expression as a manifestation of *simultaneous integration of algebraic thinking and sustainability issues* emerging in the pupils' conversation. Here, in the line of reasoning among pupils, pupil 6's expression seems to point to a connection between the materials of the toy/things, their durability and their value in the currency of carbon footprint, to add to the relationships that other pupils are putting forward.

EXPLORING CENTRAL DIMENSIONS OF DILEMMAS

Our conceptualization of dilemmas in an AWP takes its departure in Davydov's (2008) notion of *contradiction*. That is, the dilemma is seen as a *built-in incompatible condition* consisting of some kind of *value conflict* between two incompatible values. It is the *value conflict* that can invite pupils to explore, analyze and reflect on *corresponding relationships between quantitative data* in a model of reality. However, to invite the pupils to act on the value conflict, it is also important that it is *pupil-oriented* and sometimes it is necessary for the *teacher to guide* the pupils' awareness to the value conflict *by actions* such as questions or statements.

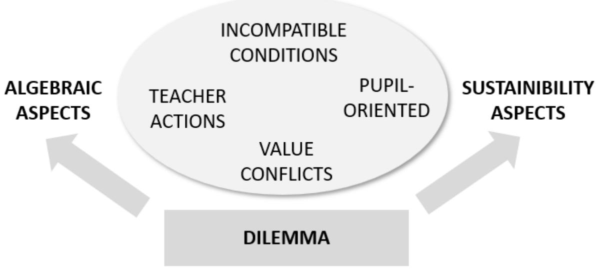


Figure 4: Central dimensions of dilemmas

In Example 1, *the built-in incompatible condition* consisted of a *value conflict* between the quantity of Lego bricks and their economic value and its carbon footprint value. The pupil-orientation was two-dimensonal. The first dimension was how, in the experience of these Swedish pupils in age 7-9 often want Lego as a gift for Christmas or for their birthdays. In the context of these children, Lego is a toy/thing that they are familiar with and that they desire. The second dimension was the economic one that Lego is expensive to buy, but it has a lower carbon footprint value with respect to other toys/things in the table. However, when the pupils were exploring, analyzing and reflecting on potential *relationships of corresponding values* they were only focusing on the Lego's economic value and its size in terms of amounts of bricks — one brick or a set of bricks. In other words, the pupils' reflections did not involve a consideration on sustainability. The conversation was left unchallenged. In this case, there could have been the opportunity for the teacher to ask questions or make statements that directed the pupil's awareness also towards the Lego's carbon footprint value and thereby to a consideration of sustainability.

In Example 2, *the built-in incompatible condition* consisted of a *value conflict* between the lol-doll's, the Lego's and the scooter's value in terms of carbon footprint and their economic value. This example also had a two-dimensional pupil-orientation. First, pupils age 7-9 in Sweden often play with this kind of things/toys and they are also aware of their economic value. Second, the conflict between the economic and the carbon footprint value is present in, for example, the cheap economic value of lol-doll in contrast to its high carbon footprint value, or a scooter that is expensive to buy but has a low carbon footprint value. The *value conflict* in this example together with the *teachers question* "Does the table feel fair?" invited the pupils to explore, analyze and reflect on *corresponding relationships* between the things'/toys' values in the table. The corresponding relationships that the pupils were establishing and reflecting on involved a consideration for *sustainability* concerning the durability of different materials to assign the things'/toys' values.

In our exploration of the situations that unfolded in the classroom, we recognized that there emerged dilemmas as pupils thought algebraically to establish connections between quantities, objects and their economic and carbon footprint values. The dilemmas were related to the built-in incompatible condition as causing value conflicts, which made possible for algebraic thinking to unfold and connect significantly to sustainability.

Our analysis of the situation allows to distinguish that the dilemma and the value conflict are connected through the incompatible conditions that are built in the carbon footprint table (Figure 2). The incompatible condiction concerns how the values of the different things/toys in the table are incompatible within one another in terms of economic value. By using a table as a model of reality within a sustainability pupil-oriented context, containing value conflicts, algebraic thinking can become mobilized as an analytical resource or tool to explore, interpret, and reason about corresponding relationships between quantities in the given model (e.g. Coles & Ahn, 2022, see also Fred et al., 2023a, 2023b). Also, the use of C-rods to represent the values seemed to be work as schematics or psychological tools which allowed the pupils to argue and reason about indeterminate quantities analytically, by using a visible and concrete artefact. This has been already documented as an important aspect of promoting algebraic thinking (e.g., Dougherty, 2008).

CONCLUSION

So far in our work, we have started the work of imagining and formulating other types of algebraic problems that invite children to connect in significant ways early algebra (grade 1-3) and sustainability. We have proposed the notion of AWP as we have invented, designed, planned, and reflected on the characteristics of situations that develop pupils' algebraic thinking to "empower children's critical reflection on current, socially relevant issues such as climate change [sustainability], and contribute to a democratic participation in communities of peers and in society" (Fred et al., 2023b, p. 2; see also Hauge & Barwell, 2017). We have found that this is not an easy task! In this paper we have suggested that dilemmas can be a way of working with this challenge. We have also visualized the multidimensional character of dilemmas by pointing at some central dimensions that we have identified so far. Thinking about how situations may be designed around built-in incompatible conditions is an initial guide to create new AWPs. In our further work with the notion of AWP we are interested in inducting new situations and doing more try-outs with pupils to deepen the understanding of dilemmas.

AKNOWLEDGEMENT

This paper reports on research conducted within the graduate school Relevancing mathematics and science education (RelMaS), funded by the Swedish National Research Council (VR3100-2908).

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VI

Paper 6: Integrating the socio-ecological and early algebra

Fred, J., Valero, P., & Van Steenbrugge, H. (2025). Integrating the socio-ecological and early algebra. In K. le Roux, A. Coles, A. Solares-Rojas, A. Bose, C. P. Vistro-Yu, P. Valero, N. Sinclair, M. Makramalla, R. Gutiérrez, V. Geiger, & M. Borba (Eds.), *Proceedings of the 27th ICMI Study Conference Mathematics Education and the Socio-Ecological* (pp. 464–471). MATHTED – ICMI.

This paper is an invitation to envision innovative directions for mathematics education, specifically within the realm of early algebraic thinking in socio-ecological contexts.

The following research question was addressed: In which ways in early algebra —and its emphasis on the development of small children's reasoning on patterns and mathematical structures (e.g., Cai & Knuth, 201)— can be linked to the socio-ecological?The findings comprise five identified features, which are linked to AWP's potential to integrate early algebra and the socio-ecological:

- AWPs involve powerful algebraic ideas sociological speaking
- AWPs mobilize a commitment with sustainability in its intellectual, ethical, and practical aspects
- AWPs have inbuilt dilemmas and value conflicts
- AWPs involve schematic, psychological, and sociological tools as materializers
- Teachers' actions promote students' engagement, exploration, and awareness

ICMI Study 27
MATHEMATICS EDUCATION AND THE SOCIO-ECOLOGICAL
Quezon City (Philippines) 22-26 January 2025

INTEGRATING EARLY ALGEBRA AND THE SOCIO-ECOLOGICAL

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This paper is an invitation to imagine new futures for mathematics education in the context of early algebraic thinking in the socio-ecological. The issue at hand is to re-imagine the concerns of why, what and how to teach central areas of the school mathematics curriculum. In this paper, the research revolves around Jenny Fred's current PhD work, in which a research team consisting of in-service teachers, pre-service teachers, teacher educators and researchers have been working on the idea of what we call algebraic wicked problems (AWP) to simultaneously integrate early algebra and the socio-ecological into students' work (Fred et al., 2024b). We highlight five pedagogical features of AWPs and the research team's work in constituting a critical work zone integrating early algebra and the socio-ecological. We conclude the paper with ideas to further theorize the possibilities of integrating early algebra and the socio-ecological in significant ways.

INTRODUCTION

As we all, even young children, are confronted with the multiple challenges of sustainability and climate change, it is necessary to make explicit connections between mathematics education and the current entanglement of societal and environmental problems. Coles (2022) emphasizes that what has been called the *socio-ecological* "points to a desire to think about human and non-human together and a key belief […] is that such a perspective is necessary for a mathematics education that is relevant to the future". (p. 212). The argument that young generations need to "build tools, options, and paths to prepare […] to confront" (Solares-Rojas et al., 2022, p. 204) the challenges posed in socio-ecological crises, is a call for mathematics educators to examine the different areas of the school curriculum and find new, significant ways to connect the contents of the curriculum and the socio-ecological. In particular, we are interested in early algebra as the area of school mathematics education that initiates and prepares children to what is argued to be one of the main components of the curriculum: algebra (e.g., Kieran, 2006). The question at stake here is the ways in which early algebra and its emphasis on the development of small children's reasoning on patterns and mathematical structures (e.g., Cai & Knuth, 2011) can be linked to the socio-ecological.

Such a connection, we argue, can be done in at least three distinct ways. A first way is to follow the traditional argument to justify the centrality of algebra in general: Since algebra is a fundamental element for successful mathematical learning as children progress through schooling, solid algebra teaching and learning within itself is the best mathematics educators can strive for. A good foundation in algebra will support all mathematics learning, and this in turn will offer people the possibility of using mathematical knowledge to solve different types of problems, among which are those posed by the socio-ecological. In other words, this first approach is what we have called the argument of *algebra as logic and psychological powerful knowledge* (Fred et al., 2022), or the option that doing the best early algebra focusing on the characteristics of algebraic reasoning in itself is the best form of connecting to the possibility of solving socio-ecological problems in the future.

A second way could be to connect the contents and forms of reasoning that are desirable to develop in children by *contextualizing problems in real-world situations of sustainability, climate change or environmental impact*. This would mean that, instead of figuring out, for example, the general rule for a number pattern without context or formulating relationships among the age of siblings, word problems could use situations and data related to CO_2 emissions in different countries or the recycling of food to express relations and patterns among quantities. This is the well-known tendency in mathematics education to bring areas of real-life to the attention of children through word problems, as pedagogical inscription devices to promote a change not only in cognition, but also in children's morality and behavior —e.g., see Yolcu & Kirschgasler, 2024, for the case of Black children to become good citizens. This second way has already been problematized, since often in this type of word problems the context is not important, and the problem-solving activity —although probably leading to desirable mathematics learning— does not invite children to go "behind the data" and engage in ethically reflective dialogue to discuss possible underlying structures/values of the data and the situation itself (e.g., Boylan & Coles, 2017; Barwell, 2013).

A third way that we have explored is what we could name *the integration of early algebra and the socio-ecological*. The current predicaments of sustainability and climate change and their multiple environmental, social and economic dimensions invite us to push the boundaries of the significance of early algebra for new forms of critical citizenship. Introducing the idea of integrating the socio-ecological into practices of teaching and learning early algebra, the socio-ecological is not only considered as a context of the problem. The context is rather seen as an integrated part of the students' work with problems where the algebra and the socio-ecological are co-constitutive. This way contributes to a consideration of early algebra as cultural and sociological powerful knowledge (Fred et al., 2022), a gap in early-algebra research yet to be tackled.

To discuss this third way and its potential to imagine new futures for early algebra in the socio-ecological, we start by presenting the research project which has led us to develop this third way. We will briefly highlight the projects' main characteristics and main results. We will then focus on what we see can help us think about the integration of early algebra and the socio-ecological, in an attempt to bring forward considerations to re-imagine the concerns of why, what and how to teach central areas of the school mathematics curriculum. We conclude by pointing to ideas on what we envision as further directions for this third way.

A CRITICAL ZONE FOR WORKING WITH ALGEBRAIC WICKED PROBLEMS

This research revolves around Jenny Fred's current PhD work (Fred, forthcoming). In a first stage of the PhD (Fred, 2019), Jenny explored aspects of teaching that create the conditions for students in grades F–3 to engage in algebraic work. In this first stage, Davydov's (2008) premise that carefully designed teaching can support children's exploration of underlying mathematical structures, and as a result the emergence of algebraic thinking oriented the work. Jenny studied students' qualitatively different ways of seeing pattern generalization to identify critical aspects in their learning (Fred & Björklund Boistrup, 2017). She also collaborated with in-service teachers to bring the principles of learning activities (Davydov, 2008) to establish and sustain algebraic work in the classroom (Fred, 2019). This first stage provided a solid idea on what may characterize productive early algebraic work in the classroom.

In the second stage of the PhD, Jenny enlarged her interest to explore in which ways productive early algebraic work does not only equip children with logical and psychological powerful knowledge, but also and simultaneously can make possible for those forms of knowing and reasoning to be culturally and sociologically powerful (Fred et al., 2022). The enlargement meant bringing in conversation the field of early algebra with the field of critical mathematics education (e.g., Skovsmose 1994) and the recent concern with mathematics education for critical citizenship in the time of climate change and socio-ecological crises. An important guiding idea was what Skovsmose and Borba (2004) have called *critical research* or the research exploration of "what is not the case —but what could be a possibility" (p. 211). Jenny organized a systematic collaborative work to imagine possible new situations. Jenny built a research team consisting of four grade 1–3 teachers working in Swedish elementary schools, four pre-service teachers enrolled in Swedish (mathematics) teacher education, and three researchers (the three authors of this paper) from Stockholm University. Jenny and the team of in-service and pre-service teachers engaged in collaboration through one school year, during which 10 research group meetings were closely documented and studied. The set of documentation of these meetings is the main core of data for the project. Besides, Jenny, Hendrik and Paola have collaborated in the analysis and discussion of the data material.

There are two interconnected processes that we want to highlight here. One is the research team's imagining of how early algebraic work can relate to social, political and environmental issues of relevance for grade 1-3 students' critical engagement. This has led to making together what we call *algebraic wicked problems* (Fred et al., 2024b). Another process is the work itself in the research team and how it constituted what we call a *critical zone*.

Algebraic wicked problems

Algebraic wicked problems (AWPs) have provided us a focus in studying the third way to connect early algebra and the socio-ecological. In studying what is not yet the case, we developed our thinking about AWPs. Whereas initially we loosely thought of AWPs as a pedagogical tool to facilitate moving in the critical and sociological terrains of early algebra, we have gradually come to a more principled understanding of central features of AWPs. Indeed, we have come to characterize AWPs by means of the following five key features:

AWPs involve powerful algebraic ideas sociological speaking
AWPs mobilize a commitment with sustainability in its intellectual, ethical, and practical aspects
AWPs have inbuilt dilemmas and value conflicts
AWPs involve schematic, psychological and sociological tools as materializers
Teachers' actions promote students' engagement, exploration and awareness

These pedagogical features have allowed us to further conceptualize the integration of early algebra and the socio-ecological. We come back to this point later. But first we delve into the research group's critical work that has led to thinking of AWPs as pedagogical tools and the conceptualization of AWPs.

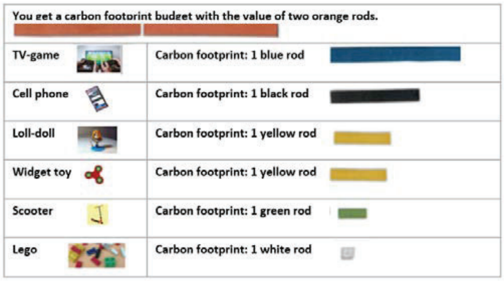
The research team's moving in a critical zone

Latour's (2018) idea of the *critical zone* has helped to conceptualize the research team's working with AWPs. According to Gleason (2019, p. 2018), the critical zone is an arrangement "where debate and conflict are necessary, where interpretive battles are waged, fueled by investments and interests, and where scientists must learn to innovate by forging new alliances between human and non-human agents, rather than speaking for the natural world". We see here connections with Coles' (2022) argument to connect the human and non-human in thinking about the socio-ecological and future mathematics education. In other words, the connection between human and non-human has helped to explore the simultaneity of early algebra and the socio-ecological.

The research group's movements during the one-year process of trying to imagine and experiment with AWPs and the connections taking place in these movements can be seen as working towards a critical work zone in which in-service teachers, pre-service teachers and a researcher-teacher educator (Jenny Fred) get the opportunity to generate and appropriate knowledge as active agents, as the AWPs are formed. Thus, the work in the critical zone consists of multiple connections that emerge between the actors consisting of persons (human agents) and algebraic aspects and socially relevant issues (non-human agents). Furthermore, these connections move over time in the collaboration (Fred et al., 2024a). Our developed methodological approach focused on the forged connections between the persons and ideas during the course of research group meetings. The methodology allowed us to trace the work in the critical zone of working with AWPs during the course of one school year. For instance, the research group's (in- and pre-service) teachers had to move away from an initial focus on students' learning of algebra, on learning algebraic concepts and linking these concepts to one another (Fred et al., 2022, see also Cai & Knuth, 2011, Kieran, 2018) to instead moving to focus on cultural and sociological interpretations of algebraic work concretely tied to daily classroom practice.

THE SIMULTANEITY OF ALGEBRA AND THE SOCIO-ECOLOGICAL

So far, developing AWPs in a critical zone of collaboration has led us to identify five pedagogical features that we see are connected to the potential of AWPs to integrate early algebra and the socio-ecological. First, the pedagogical feature *powerful algebraic ideas sociological speaking* allows us to think about algebra as a resource or tool for critically evaluating models such as for example diagrams, graphs of different kinds of socio-ecological concerns. The integration of algebraic ideas and the socio-ecological becomes realized, when algebra is used as an analytical resource, tool or facilitator, for concrete critical thinking and action, when exploring, interpreting, analyzing and reasoning about the underlying structures of, and possible corresponding relationships between, given values or quantities of data. This kind of work can also enable students to become aware of the limitations of such models, which only visualize certain "things" and leave others unnoticed (Fred, et al., 2022; Fred et al., 2023; 2024b). Second, the *commitment with sustainability in its intellectual, ethical, and practical aspects* (see Öhman & Sund's (2021) didactic model of sustainability) allows us to think of sustainability not only as a context, as in the case of the school tradition of word problems, that surrounds the problem (Fred et al., 2023). Instead, the different commitment aspects of sustainability can help us to think of the context in a broader sense and thus enable a simultaneous integration of algebra and the socio-ecological. The intellectual aspect relates to the selection of subject content on sustainability. In making this selection, it could be essential to consider that the subject content is engaging and relevant to students' lives. In other words, it may be important that the content chosen is dependent on the age of the students, their geographical location and their current concerns about sustainability issues. It could be a topic that is frequently addressed in the media or in the local communities. For example, in Sweden "Lilla Aktuellt", a television news program for children which students often watch at school, may be a source of information. Related sustainability news topics could then become a focus to work around. Another illustrative example related to intellectual aspects is the carbon footprint table (Figure 1), constructed and used in our try-outs with students in a classroom context.



You get a carbon footprint budget with the value of two orange rods.	
TV-game	Carbon footprint: 1 blue rod
Cell phone	Carbon footprint: 1 black rod
Loll-doll	Carbon footprint: 1 yellow rod
Widget toy	Carbon footprint: 1 yellow rod
Scooter	Carbon footprint: 1 green rod
Lego	Carbon footprint: 1 white rod

Figure 1: A carbon footprint table

The table lists things/toys and their carbon footprints values expressed in terms of Cuisenaire rods. When asked to choose what they wanted to buy with a budget of two orange rods, students started to discuss the given values in terms of durability of different materials (Fred et al., 2023). The intellectual aspect could here be understood as the knowledge of durability of different materials. The integration of algebra and socio-ecological consists here in the sense of exploring, analyzing and reflecting on the underlying structures of the given values to establish and reflect on a consideration for sustainability in terms of the durability of different materials to assign values to the things'/toys' (Fred et al., 2024b). The emotional aspect addresses the choice of data in the model of reality, which prompts students to interact with one another and challenges them to engage in critical reflection and discussion. The aim for students is to become emotionally involved, which may challenge some of their assumptions about how they or their family live, which then also connects to an ethical dimension that requires careful handling and respect. The ethical could be put in relation to, on the one hand, the desire to buy a toy one likes and, on the other, the responsible consumption of the environmental recourses on Earth. One related example that came up in the research team's reflection is when children started to discuss the desire for having a new mobile phone in contrast to keeping the used/refurbished one with respect to the consumption of the environmental resources on Earth (Fred et al., 2023). In this situation, the research team imagined a not yet tried-out problem situation, where students would be invited to reason about three fictional persons and their budgets, where the value of a mobile phone should have different values depending on the person's environmental choices, e.g., if one prefers to buy a new or a used/refurbished telephone. This was suggested by the in-service and pre-service teachers as a possibility to invite children to explore, interpret and reason about how the use of Planet Earth's resources relates to emotional/ethical aspects in the different budgets. The practical aspect focuses on how the subject content and the choice of data can prompt students to act. The actions undertaken in students' work are not limited to practical tasks such as sorting waste or establishing an environmental group. Rather, they encompass a broader range of activities, including discussions, reflection, and the formulation of new ideas (Öhman & Sund, 2021).

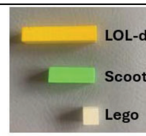
Third, the pedagogical feature of *having inbuilt dilemmas and value conflicts* takes its departure in Davydov's (2008) notion of contradiction and allows us to think of how to challenge the students to explore, interpret, analyze and reason algebraically about the underlying structures of, and possible corresponding relationships between, given values or quantities of data. That is, the dilemma is a built-in incompatible condition consisting of some kind of value conflict between two incompatible values. The value conflict invites students to explore, analyze and reflect on corresponding relationships between quantitative data in a model of reality. In our exploration of the classroom situations, we recognized that there emerged dilemmas as students worked algebraically to establish connections between quantities, objects and their economic and carbon footprint values. The dilemmas were related to the built-in incompatible condition causing value conflicts, which made it possible for algebraic thinking to unfold and connect significantly to sustainability.

Fourth, the pedagogical features, *schematic, psychological and sociological tools as materializers,* can facilitate students' argumentation, reasoning, and exploration of underlying structures and values that are of a socio-ecological nature. In the try-outs, the use of C-rods as representing the values of objects seemed to allow the students to argue and reason about corresponding relationships analytically, by using a visible and concrete artefact. The C-rods mobilized the conversations. More specifically, they were using the artefact when reasoning about the value conflict between the toys' and things' economic and environmental values, as illustrated below:

Student 4: That little lol-doll can't be much more than Lego, can it?

Student 5: And then the scooter... Scooters are big...

Student 6: But you have a scooter for a really long time... And there are a lot of other things than just plastic.



Fifth, the pedagogical feature *teachers' actions promote students' engagement, exploration and awareness* allows us to think of how the teacher sometimes needs to guide the students' awareness to the in-built dilemmas, the incompatible conditions, or the value conflict by actions such as questions or statements. In other words, the teachers' actions become crucial to enable the integration between aspects of algebra and the socio-ecological. In one of the try-outs, a teacher's question "Does the table feel fair?" together with the value conflict seemed to invite the students to explore, analyze and reflect on corresponding relationships between the things'/toys' values in the table and a consideration for sustainability concerning the durability of different materials. In another instance unfolded by other questions, had no challenge. The value conflict that was realized concerned the quantities and their correspondence in value with no integration of socio-ecological:

Student 1: What is one [emphasizes is and one] Lego?

Teacher: This is probably a box of Lego. Not just a piece but... What do you say? What do you think?

Student 2: One Lego bag?

Teacher: One Lego bag? And...

Student 1: I think it is one Lego set.

Teacher: Yes, one Lego set. But not the biggest.

The actions of the teacher are of great importance, as they facilitate the students' awareness of both the algebraic and socio-ecological aspects of the subject matter. Furthermore, they encourage the students to persevere in their endeavors, rather than merely attempting to identify a single optimal solution, which they typically are used to.

DISCUSSION AND QUESTIONS FOR THE FUTURE

After developing our initial thinking about what AWPs may be, we conclude that the five pedagogical features highlighted above are a way to make visible the elements that come together in the attempt of designing teaching where both early algebra and the socio-ecological are equally significant and become integrated. Towards the future, there emerges the question of how to better conceptualize AWPs, their characteristics and the types of activities and relationships that may take place among different actors. Following Latour (2018), we have started to pay attention not only to the human actors involved (e.g., the teacher, students) but also to the non-human actors (e.g., algebra, socio-ecological problems, models, Cuisenaire rods, etc.) that become entangled as an AWP situation unfolds. If AWPs is to become an idea for early algebra teaching and learning in the socio-ecological, we would not want it to be reduced to or become conflated with a nice, new world problem. Rather, AWP can be complex and unpredictable situations where these multiple actors mobilize, enter in relationship and generate new possible agencies together. In other words, they are dynamical spaces for action, a network of human and non-human actors gaining agency to make possible the integration of algebra and the socio-ecological (see e.g., Fred et al, 2024a).

We would also like to conceptualize algebra and what may be early algebraic work as potentially dynamic agents. In contrast to the view of algebra as a "dead" content of the school curriculum, the finding of underlying structures (relationships, variables and constants, and the patterns among them) come to life as a potential actor in exploring the multiple value dilemmas in socio-ecological predicaments. This type of dynamics was clearly visible in the example presented above. The challenge of conceiving of early algebra and what it can do as a force in AWP is an important point towards advancing the dominant view that has been established in the field concerning algebra as powerful logical and psychological tools. Furthermore, we also want to initiate a discussion about the notion of critical zone as a possible methodological approach within mathematics education research that allows the imagination of new knowledge, curricula and pedagogies through the interconnection of the human and the non-human in future mathematics education practices. In other words, to create new cross-contextual spaces for imagining new relations of greater mutuality across multiple scales of space, time and concepts.

Acknowledgements

This research is part of the graduate school RelMaS: Relevancing Mathematics and Science Education (VR 2019-03679), funded by the Swedish Research Council.

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Chapter 6 – Characterizing Algebraic Wicked Problems (AWPs)

In this chapter, I present the most salient result of my doctoral work, which emerged from addressing the following two research questions: How can we understand algebraic ideas for the combined purpose of developing younger students' algebraic thinking and fostering citizenship? What characterizes activities that integrate both algebraic and socio-ecological aspects?

I propose the notion of Algebraic Wicked Problems (AWPs) to characterize a type of problem situation in which early algebra and socio-ecological aspects are equally significant (Paper 5). These problem situations are critical—that is, they involve critical features in both the LAT and CME senses. Furthermore, there is a desire to integrate the algebraic and socio-ecological aspects simultaneously into students' work with AWPs (papers 4, 5, and 6). In this conceptualization, the socio-ecological components are a fundamental ingredient of AWPs, rather than merely a context surrounding the algebraic content. The idea of AWPs emerged from the work presented in chapters 3, 4, and 5.

AWPs stand in contrast to the common view of problem-solving in school mathematics, which holds that, despite following different approaches, students are expected to arrive at a solution. For example, Marchant et al. (2022) describe reaching a solution as fundamental to problem-solving in school mathematics. Instead, AWPs encourage students to persevere in their endeavors rather than merely seek a single optimal solution. More importantly, students learn that addressing a wicked problem demands an appreciation of its complexity and insolvability. Yet, different discussions and choices are possible, and this recognition is more important than reaching a solution. Thus, the framing of AWPs comprises multiple layers that extend beyond merely providing students with a model, diagram, or table. Moreover, AWPs promote an open learning environment in which the teacher adopts a responsive, facilitative role. This highlights the significance of the student–teacher relationship.

The following dimensions, which I describe in more detail in subsequent sections, are proposed as characteristics of AWPs:

- AWPs involve culturally and sociologically powerful algebraic ideas
- AWPs involve critical situations within students' life-worlds
- AWPs involve different dimensions of abstraction
- AWPs involve built-in dilemmas and value conflicts
- AWPs involve schematic, psychological, and sociological tools as materializers

AWPs involve culturally and sociologically powerful algebraic ideas within students' life-worlds

AWPs involving culturally and sociologically powerful ideas concern the application of algebraic ideas in young students' life-worlds, such as algorithms, mathematical models, and algebraic concepts. This characteristic supports the contextualization of an AWP by integrating powerful algebraic ideas into a socio-ecological concern, thereby serving as a foundation for students' exploration and critical reflection. This kind of contextualization, therefore, can enable critical conversations that address the power structures inherent in a society in which powerful algebraic ideas are situated.

The above-described integration can be realized when students explore corresponding relationships between quantitative data in diagrams, graphs, tables, or other models of reality, and point out possible corresponding relationships in terms of socio-ecological values, such as, for example, minimizing one's carbon footprint and desire, and responsible consumption of the Earth's environmental resources. One illustrative case involved students interpreting and reasoning about which factors influenced the given value of the electric scooter, the LOL doll, and Lego in the carbon footprint table (see papers 4, 5, and 6).

Early algebra becomes a powerful modeling language for expressing abstract socio-ecological structures and their relationships. It encompasses a cluster of modeling languages, both within and beyond mathematics, comprising symbols (such as letters), variables, and unknowns. It directs attention to how these elements can be used to express, reason about, and model underlying structures and relationships. Furthermore, it relates to how schematization, visualization, and symbolic representation, through modeling languages, may facilitate the identification of underlying structures, patterns, and relations.

A similar approach to contextualization involves inviting students to engage in reasoning about powerful algebraic ideas as performative agents that do not decide who or what is to be served. This also includes conversations concerning the interrelations between mathematization and demathematization as components of a cohesive process.

AWPs involve critical situations within students' life-worlds

The AWPs involving critical situations within students' life-worlds, a notion from CME, can be seen as an extension of the concept of problem situations in relation to Davydov's work and LAT conceptualization. The extension lies in highlighting which elements, in terms of contextualization within AWPs, encourage students to integrate early algebraic and socio-ecological aspects into their work. In general, they are scenarios that provide contexts in which algebra becomes real. It supports the consideration of contexts that encourage students to engage in algebraic activities while integrating socio-ecological aspects.

As argued in Paper 4, the contextualization phase within AWPs encompasses multiple layers that extend beyond merely presenting students with a mathematical model. In Chapter 5, I highlight three identified layers: the first concerns students' everyday relations and experiences in their life-worlds; the second addresses a socio-ecological issue—a critical situation with tangible and significant impact on people; and the third emphasizes critical citizenship, focusing on students' reflective engagement with socio-ecological aspects such as consumption in relation to the resources of our planet. These three layers together invite students to engage in algebraic activity in which they explore, identify, and reflect on the underlying structures of objects and on possible relationships between those objects that are of socio-ecological character.

Critical situations address the images of reality that might be represented within a mathematical model. In Chapter 4, the critical situation of carbon dioxide emissions—a situation that affects people in concrete, serious ways—was addressed. This situation seems to invite students to express an ethical awareness of the relationship between consumption and our planet's resources. It also brings to light possible hidden features or values relevant to students' life-worlds that may be present in the mathematical model. We argued in Paper 5 that features or values should refer to the everyday relations and experiences students inhabit, in which different values or features are situated. This can also be pertinent to the mathematizing and demathematizing that are part of daily life. Citizenship addresses societal considerations of the ethical, political, social, and ecological features and values. In Paper 6, we discussed how data in a model of reality can encourage students to interact, critically reflect, and discuss topics such as the desire to buy a preferred product versus responsible consumption of Earth's resources. This pedagogical feature supports the setting of the scene for students to explore the relationships among thinking abstraction, realized abstraction, and real abstraction. Such an exploration can allow for a deeper understanding of how mathematics influences society, enabling students to recognize the impact of mathematics on decision-making and action, skills vital for fostering critically engaged citizenship. The wicked nature of a critical situation helps build a sense that mathematical tools do more than find effective solutions. They can also help, as Haraway (2016) says, to "stay with the trouble".

AWPs involve different dimensions of abstraction

The fact that AWPs involve different dimensions of abstraction means that the critical problem situation invites students to relate thinking abstractions, such as mathematical models, to realized abstraction. The results of papers 5 and 6 highlight how students engaged in exploration, interpretation, and reasoning to ascertain which object characteristics influenced the values presented in the carbon footprint table. This process required integrating a thinking abstraction—the carbon footprint table—and a realized abstraction, as students reflected on the assigned values and considered the objects' materiality in the table. They also investigated the formatting power of mathematics by examining the interplay between the three forms of abstraction, such as how durability affected the assigned carbon-footprint value, for example, by noting that an electric scooter can be used for an extended period.

This means that students might be invited to structure features within each abstraction and relate these structures to one another. First, in thinking abstraction students might structure the features of a mathematical model or concept. Second, in realized abstraction, students might then recount what these features represent—in other words, what might then be perceived as reality.

The students' reflections could, for example, concern how mathematical concepts, such as mathematical models, are constructed as representations of reality or a phenomenon. This is also linked to the development of skills in discerning and critically reflecting on the formatting power of mathematics, thereby fostering critical citizenship. Through engaging with and organizing features across these two abstractions, students gain not only knowledge of algebra and abstraction but also a deeper understanding of how mathematics shapes society. This enables them to appreciate the influence of mathematical models and concepts on decision-making and action—skills essential for becoming critically engaged citizens.

This idea highlighted aspects that reflect the underlying reality represented by mathematical concepts and models, emphasizing their potential for effective contextualization. Models inherently contain elements of the reality that they emphasize, which serve as a form of contextualization. Therefore, it highlighted the significance of the internal structure of mathematical models, including their "hidden" features. The term "hidden" refers to the fact that mathematical models are abstractions that present only selected features of a given reality. By understanding the internal structure, one can identify which features have been selected and, consequently, discern what aspect of reality the mathematical model represents. This directed attention to the significance of the chosen features within a mathematical model and the aspects of reality they may represent.

AWPs involve built-in dilemmas and value conflicts

The built-in dilemmas and value conflicts not only mobilize algebraic aspects that come into play in students' work but also integrate them with socio-ecological aspects. They mobilize abstract structures and relationships in critical situations and can therefore use the formatting power of mathematics to call attention to the need for a critical awareness of the "invisible" power that mathematics can exercise.

In the students' work, a dilemma arises from an inherent incompatibility, typically involving a value conflict between two opposing values. It is this value conflict that can invite students to explore, analyze, and reflect on corresponding relationships between data in a model of reality. However, it is also important that the invitation to students to act on the built-in dilemma is student-oriented. For instance, in the carbon footprint table illustrated in Figure 8, each item is represented by its carbon footprint value rather than by the conventional economic value that may be more familiar to students. Additionally, values assessed by carbon footprint do not necessarily align with the items' financial value; an object deemed expensive from an economic standpoint may, in fact, have a comparatively low carbon footprint. Additionally, the teacher must sometimes guide students' awareness toward the value conflict through actions such as questions or statements.

In Paper 5, it is argued that the dilemma and the value conflict are interconnected through the incompatible conditions embedded in the carbon footprint table (see Figure 8), which are represented by using Cuisenaire rods (hereafter referred to as C-rods).

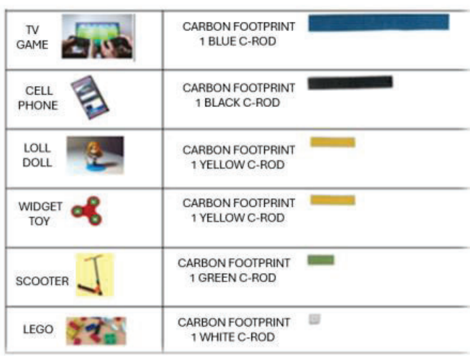

Figure 8. The carbon footprint of an AWP tried out with students in grades K-3

These incompatibilities arise from the economic incongruence between the values of various items or toys in the table. Employing such a table as a model of reality within a sustainability-focused, student-oriented context—where value conflicts are present—enables the mobilization of algebraic thinking as an analytical tool for exploring, interpreting, and reasoning about relationships between quantities in the given model.

The above suggests that this pedagogical feature is multidimensional. Therefore, when designing for an incompatible condition, one needs to examine the possible hidden features, values, and assumptions that may arise from it. For example, students' exploration of and reflection on the formatting power of mathematics can be realized through value conflicts. Value conflicts can invite students' reflections on values that are not purely mathematical but relate to people or the environment, and that may also be embedded in the ways mathematics operates and serves different purposes. These values are connected not only to the internal logic of mathematics but also to important socio-ecological values. They can mobilize an exploration of abstract structures and relationships in mathematical models of critical situations and draw attention to the need for critical awareness of the "invisible" power that such models can exercise.

Furthermore, these built-in dilemmas and value conflicts can help students make connections among thinking abstraction, realized abstraction, their relationships, and the formatting power of mathematics. For example, during the tryout, a value conflict arose over a toy that was cheap to buy but had a high carbon footprint in the table. This value conflict invites students to establish connections between quantities, objects, and their economic and carbon-footprint values. The dilemmas stemmed from the built-in incompatibility, which created value conflicts and enabled algebraic thinking to unfold and connect more closely with sustainability.

The teacher's question plays a special role here: it serves as a didactic "opener" that shifts the work from merely reading the table values to exploring and reflecting on their rationality and on identifying possible underlying variables, thereby making visible the formative power of mathematics. For example, when students link sustainability/durability to a given carbon value, the relationship among different forms of abstraction (and their consequences) becomes something that can be jointly investigated.

AWPs involve schematic, psychological, and sociological tools as materializers

AWPs involving schematic, psychological, and sociological tools to support the materialization of arguments, problems, and reflections can be seen as an extension of the concept of learning models in relation to the work of Davydov and Eriksson and colleagues (2024), emphasizing learning models as a means of materializing algebraic thinking in joint actions. The extension lies not only in viewing these tools as mediating artifacts for visualizing and exploring abstract structures within algebraic objects or for materializing algebraic thinking, but also in engaging with dynamic abstract structures that exist within critical situations in students' lifeworlds. These dynamic abstract structures are sociological, and include social, cultural, ethical, political, and ecological dimensions. These learning models can serve as materializers of value conflicts and as contexts for students' exploration, identification, reflection, and argumentation when relating to and structuring features in action. Since value conflicts are inherently abstract and may not be readily apparent to students, a sociological tool can serve as an effective mediating artifact, facilitating reflection on the underlying features that constitute such value conflicts.

These tools can also materialize students' argumentation, reasoning, and exploration of abstract socio-ecological structures and values. In the tryouts in the empirical setup, the use of C-rods to represent the objects' values appeared to enable students to engage in analytical argumentation regarding corresponding relationships, using a visible, concrete artifact. The C-rods mobilized productive conversations; specifically, students used the artifact to reflect on the conflict between the economic and environmental values of toys and other objects.

In papers 4 and 5 we argued that using C-rods, as concrete representations of object values, helped students analytically reason about corresponding relationships. The C-rods facilitated discussion, especially around conflicts between economic and environmental values in toys and objects. These C-rods can also support the materialization of relationships between two abstractions, thereby recognizing the formatting power of mathematics.

Summarizing comments

Overall, AWPs can be understood as a way of operationalizing and integrating three lines of research addressed in Chapter 4 as central to orienting what and how early algebra can be taught, while also integrating aspects of socio-ecological crises. The first concerns the expansion of Davydov's concept of learning models. In AWPs, Davydov's concept of learning models is integrated with socio-critical modeling (e.g., Barbosa, 2006; Coles, 2023b; de Freitas et al., 2022; Solares et al., 2022) by addressing how schematic models, tables, diagramming, and concrete artifacts can be used not only to visualize mathematical relationships but also to materialize values, interests, and conflicts that belong to students' life-worlds. Second, the notion of AWP draws on research on mathematical models in critical situations and the formatting power of mathematics (e.g., Kwon et al., 2021; Skovsmose, 1994; Steffensen & Karceja, 2021). AWPs address how the relationship between the formatting power of mathematics and the interplay among different forms of abstraction can be connected in younger students' work, while also examining critical socio-ecological situations within their life-worlds. In relation to this, AWPs give an idea of how early algebra can become situated within this complex network of interconnections. Third, AWPs is based on research on the notion of wicked problems and on how it can be used in teaching (e.g., Rittel & Webber, 1973; Block et al., 2019; Kynigos, 2024; Löngren & van Poeck, 2021; Steffensen et al., 2018), but it moves this type of problem into an early algebraic practice, where the role of early algebra also becomes a resource for developing a sensibility toward the complexity of socio-ecological issues. This allows, as Haraway (2016) says, to "stay with the trouble".

In Chapter 7, these results serve as a foundation for a concluding discussion that shifts the focus from the characterization of AWPs to consequences: in terms of the epistemological dimensions of knowing, including how diverse domains of knowing are integrated and interrelated, and how LAT and CME together provide a basis for understanding and designing such activities. Thus, the characterization of AWPs serves as a bridge for discussing both theoretical implications and future directions—how teaching in early algebra can be developed to simultaneously support algebraic thinking and an emerging critical citizenship in a contemporary socio-ecological world.

Chapter 7 – Concluding discussion

While Chapter 6 presented the empirical contribution of the thesis by characterizing AWPs, this concluding discussion draws together the central theoretical contributions by addressing, in an integrated way, the what, how, and why of early algebra in the context of critical socio-ecological situations in young students' life-worlds.

The powerful complementarity of LAT and CME

At a theoretical level, the thesis contributes by bringing together, and using complementarily use, the tools of LAT and CME. This need emerged from the overarching concern of how to connect a) the knowing of early algebra and critical socio-ecological situations, b) how young students can come to know early algebra in a complex network of interconnections among mathematical, cultural, historical, social, political, and ethical concerns, and 3) why such knowing matters in relation to contemporary socio-ecological challenges and questions of life in and with the world.

These two types of theories usually point in different directions. LAT is frequently used in research that explores the development of teaching and learning, research that examines the details of what Skovsmose and Valero (2008) call powerful mathematical ideas in "logical" and "psychological" terms, whereas the tools of CME are used in research exploring powerful mathematical ideas in "cultural" and "sociological" terms. Because these perspectives allow one to view powerful mathematical ideas through different lenses, they are not often brought into dialogue. In this chapter, I argue that placing LAT and CME in productive tension makes it possible to conceptualize how mathematical ideas can become powerful not only logically and psychologically, but also culturally and sociologically. Together, they provide powerful theoretical tools for teachers and researchers to continue imagining activities that challenge young students to integrate early algebraic and socio-ecological aspects simultaneously in their work. This enables new ways of teaching early algebra to young children that are relevant and up to date, and that question the function of mathematics education in schools, engaging people with problems of significance for themselves and the world.

When they are connected to characterizing AWPs and are concretized as didactic tools, the concepts in LAT and CME together create opportunities to invite students into algebraic activity that gives equal space to aspects of early algebra and to socio-ecological crises within students' life-worlds. The power of these complementary tools lies in situating early algebra within a complex network of interconnections among mathematical, cultural, historical, social, political, and ethical concerns.

Reconceptualizing early algebra in the socio-ecological

The second part of this thesis, following the licentiate thesis, started with the overarching question of how early algebra could be important not only for students' mathematical development but also for students' role as critical citizens in a world facing socio-ecological crises. The intention was to find alternatives for teaching and learning in which early algebraic thinking and socio-ecological aspects become interdependent and inform one another. Here, I propose a final characterization of early algebraic work that responds to this concern.

Knowing algebraically in critical situations

Extant research literature proposed that early algebraic thinking involves students' engagement in activities such as noticing structures (Kieran, 2006), investigating underlying mathematical structures (Davydov, 2008), exploring and identifying patterns (Mason, 2018), expressing algebraic generalizations (Radford, 2006), and drawing general conclusions (Carraher, Martinez et al., 2008).

When working with AWPs, young students' early algebraic activity is interpreted through the CME perspective and in relation to ongoing socio-ecological crises, broadening what engagement in early algebraic activities can entail. In this respect, the focus shifts from treating algebraic activity as primarily decontextualized work with general structures (e.g., recognizing number patterns) to understanding algebraic knowing as enacted within contextually embedded practice. Accordingly, the meaning and direction of students' algebraic activity are reconfigured when what is at stake includes socio-ecological relations and consequences. Thus, the conceptualization of school algebraic knowing goes beyond its conventional scope by foregrounding how algebraic thinking can be oriented toward—and shaped by—critical situations in students' life-worlds. Young students' early algebraic knowing becomes, in this sense, intertwined with critical situations within their life-worlds. Therefore, the contextualization phase of AWPs is crucial because it is central to mobilizing knowing algebraically in critical situations.

Within this framing, knowing extends beyond understanding general structures, such as recognizing number patterns, and encompasses interactions that are connected to their life-worlds. Algebraic knowing is positioned within an interactive web of interconnections rather than as an isolated set of skills. In this web, algebraic knowing comes to life through relating and structuring that take place between: a) early algebra and mathematics in action, b) early algebra and the formatting power of mathematics, and c) socio-ecological crises and shaping young students' life-worlds. Coles et al. (2023b) claim that incorporating a socio-ecological crisis into mathematics education entails foregrounding critical situations in which mathematical abstractions and socio-ecological issues such as climate change, pandemics, sustainability, and humans' impact on non-human nature meet inseparably. Therefore, early algebraic knowing is interwoven with values, such as sustainability-related values, that extend beyond algebraic content to include young students' life-worlds. Such values are not determined solely by the teacher in advance; indeed, students' conversations may address various socio-ecological values bringing in ecological, social, political, or ethical dilemmas into early algebraic reasoning. This aligns with the socio-ecological turn (le Roux et al., 2025) and Coles' (2022) argument that the socio-ecological crises themselves become an integral part of students' mathematical knowing. Rather than framing school algebra as the mastery of algebraic objects, knowing in critical situations emphasizes dynamic, contextually situated engagement. In AWPs, early algebraic knowing is therefore inseparable from the practices and situations with resulting, and sometimes conflicting values in which it is enacted, and from learners' lived experiences of mathematics in action. This orientation highlights the civic dimensions of mathematics education by foregrounding how algebraic activity can address issues arising in young students' life-worlds (papers 4—6).

Relating and structuring in action

The early algebra addressed in Chapter 2 foregrounds the problematization of algebraic thinking, focusing on identifying and expressing relationships, patterns, structures, and generalizations. When young students work with AWPs, they are relating and structuring data in action: they identify underlying abstract structures, organize the data using these structures, and create relationships between the data to make meaning and find connections. The carbon footprint table from papers 4, 5, and 6 exemplifies mathematics in action. As in Kwon et al.'s (2021) analysis of how mathematical representations used in public discourse shape readers' perceptions, the carbon footprint example shows that learning early algebra in action involves relating thinking abstraction to realized abstraction and, at times, to real abstraction. This requires students to structure each abstraction and connect these structures: a) articulating key features of a model or concept (thinking abstraction), b) relating these features to what they represent (realized abstraction), and c) considering how such representations shape decisions and actions (real abstraction).

Overall, engaging with these abstractions helps students discern and critically reflect on the formatting power of mathematics, thereby fostering the possibility of an emerging critical citizenship. As young students structure and relate between the abstract structures they have identified across models and contexts, they develop algebraic understanding alongside an appreciation of how mathematical representations inform judgment, decision-making, and action. In the previous example, the concept of formatting power is not explicitly discussed; that is, students are not prompted to reflect directly on this issue. Steffensen and Kacerja (2021) present a clearer illustration that explicitly attends to the role of mathematics in shaping perspectives. They do so by using Carbon Footprint Calculators (CFCs) to examine questions related to climate change and the role of mathematical modeling. Although their example is from work with older students, it can support thinking about how younger students can be invited to engage in similar, age-appropriate work.

This discussion also aligns with Skovsmose's (2021) account of different relationships between mathematics and crises, including mathematics picturing crises, mathematics constituting a crisis, and mathematics formatting crises. In such cases, learning early algebra can be mobilized as young students structure data within a mathematical model, deciding which variables to foreground when picturing a socio-ecological crisis. Work such as that on the carbon footprint table can therefore be extended to broader algebraic ideas, including reflection on how models and algorithms may not merely describe situations but also help shape them.

Returning to the argument above, sociological tools support students' argumentation, reasoning, and examination of complex socio-ecological structures and values. From the perspective of de Freitas et al. (2022), diagrams can be understood as materializing sociological tools that enable alternative perspectives and hypothetical scenarios; in the terms used here by giving form to emerging ideas so that they can be tested, revised, and discussed.

Coles (2023, p. 28) further explores this approach by linking it to the "processes of creative abstraction." The emphasis is not on the model or diagram as an endpoint; rather, these tools can be used to relate and structure underlying abstract structures, thereby enhancing understanding of these structures associated with socio-ecological issues. This view aligns with the arguments of Solares et al. (2022) regarding socio-critical modeling.

Orienting the gaze toward the future

The characteristics of AWPs have so far been evaluated in limited trials with students, but the identified characteristics point to a promising way of connecting early algebraic reasoning to engagement in socio-ecological crises. For educators and researchers, a key task is to further develop and exemplify these characteristics across additional topics and classroom settings, with attention to the internal structures and embedded dilemmas through which teaching and learning of early algebra connect to concerns of the world. In this way, AWPs can contribute both theoretical insight and practical approaches that support students' learning of early algebra within and beyond the classroom. The conceptualization of AWPs and their potential to engage students in integrating algebraic and socio-ecological aspects concurrently provides a valuable basis for future progress.

In conclusion, the collaborative work on writing a chapter (Abtahi et al., forthcoming) in the volume of the 27th ICMI Study is already orienting the gaze toward the future. In this chapter, I contribute to reflections in two different dimensions. The first dimension concerns shaping transdisciplinary spaces for enabling understanding activities in which diverse domains of knowing are integrated and interrelated (e.g., Chronaki, 2000). The second use of Abtahi's (2022) idea of considering the complexity of coming to know as interrelational epistemology. More precisely, a way of becoming to know things that considers learners as selves-in-relations rather than isolated individuals and knowing as an understanding of one's position in webs of inter-relationality.

In relation to the first dimension, I reflect on how AWPs invite knowing early algebra within an interactive web of interconnections, in which early algebra is just one of the actors' concerns, showing that knowing early algebra. Knowing abstract structures within a web of interconnections between mathematics in action and socio-ecological dimensions illuminates how these interconnections can be explored and reflected upon.

In relation to the second dimension, I reflect on how the teacher's pedagogical actions, in terms of the questions they pose or the tasks they assign to students, bear implications for the kinds of dynamic webs of interrelatedness that emerge. In other words, when there are changes to the structures of the interrelation web and the knowing that a student as a self comes into relation with, the teacher's actions play a crucial role.

I also see myself initiating new opportunities for collaboration in several ways. One involves initiating a research project to further explore and develop the features of AWPs. In this kind of research project, I see it as important to involve teachers and pre-service teachers, drawing on their backgrounds to foster dynamic collaborations that can imagine "things" that do not yet exist. Furthermore, I see value in broadening the idea of AWPs in relation to transdisciplinary approaches to socio-ecological crises and early algebra, as well as in relation to investigating evolving structures within these dynamic networks of interrelations.

Svensk sammanfattning

I avhandlingen undersöks hur förutsättningar kan skapas som gör det möjligt för elever att delta i algebraiska aktiviteter. Avhandlingen positioneras inom, och bidrar till, den socioekologiska vändningen i matematikdidaktisk forskning genom att undersöka hur undervisning och lärande i tidig algebra, i samspel med socioekologiska kontexter, kan utgöra en central grund för elevers engagemang i matematiska problem. I denna ansats förstås algebraiska och socio-ekologiska dimensioner som ömsesidigt konstituerande och som därigenom integreras med varandra.

Problemställning

Forskare inom tidig algebraforskning betonar vikten av att ge yngre elever möjligheter att utveckla algebraiskt tänkande (se t.ex., Kieran, 2022; Kieran et al., 2016). Förmågor som att kunna uttrycka algebraiska mönstergeneraliseringar (se t.ex., Radford, 2014), att etablera relationer och uttrycka generaliseringar (e.g, Arcavi et al., 2017, att resonera algebraiskt (Blanton & Kaput, 2008), att identifiera strukturer (Kiearan, 2006) att dra generella slutsatser (Carraher, Martinez et al., 2008) beskrivs som avgörande eftersom de kan fördjupa elevernas förståelse av olika matematiska områden (e.g., Kaput, 2008). Vidare betonas vikten av undervisning som engagerar elever i algebraiska aktiviteter inom relevanta samhälleliga sammanhang. Relevanta frågor inkluderar hur sådant arbete kan initieras och upprätthållas i klassrumspraktiken, hur undervisningen kan utformas för att stödja elevernas utveckling av algebraiskt tänkande och vilka konsekvenser som kan följa när undervisningens mål breddas till att omfatta aspekter av medborgarskap (Skovsmose, 1994) inom en socio-ekologisk kontext (Coles, 2023b). Avhandlingen syftar därför till att undersöka hur socio-ekologiska överväganden kan integreras i undervisning och lärande av tidig algebra.

Syfte och frågeställningar

Det övergripande syftet med denna avhandling är att undersöka hur förutsättningar kan skapas som gör det möjligt för studenter att delta i algebraiska aktiviteter i tidig algebras kontext. I den första delen av avhandlingen utforskas detta i relation till syften och överväganden kring elevernas fortsatta matematikinlärning. Den andra och tredje delen har ett bredare perspektiv och behandlar aspekter av medborgarskap inom en socio-ekologisk kontext. Forskningen är strukturerad kring tre övergripande frågor:

1. Vilka aspekter av undervisningen kan skapa förutsättningar för yngre elever att delta i algebraiska aktiviteter?
2. Hur kan vi förstå algebraiska idéer med det gemensamma syftet att utveckla yngre elevers algebraiska tänkande och främja medborgarskap?
3. Vad kännetecknar aktiviteter som integrerar både algebraiska och socio-ekologiska aspekter?

Avhandlingens disposition

Avhandlingen är strukturerad i tre delar. Den första delen är min licentiatuppsats, som redan har granskats och godkänts (Fred, 2019b). Den behandlar den första forskningsfrågan och presenteras i kapitel 2. Den fokuserar på att skapa förutsättningar för yngre elever att delta i algebraiska aktiviteter i tidig algebra. Den andra delen behandlar den andra forskningsfrågan och markerar en förskjutning mot ett delvis nytt problemområde inom tidig algebra. Genom att introducera Critical Mathematics Education perspektiv (CME; se t.ex. Skovsmose, 1994, 2023a, 2024) vidgas förståelsen av möjliga frågor och syften för undervisning och lärande i algebra i relation till medborgarskap. Den andra delen behandlar den andra forskningsfrågan och tar upp en expansion inom tidig algebra, när antagandet av CME (se t.ex. Skovsmose, 1994, 2023a, 2024) vidgade möjliga uppfattningar om frågor och syften med undervisning och lärande av algebra för medborgarskap. Denna del utvecklas i kapitel 3. Den tredje delen utforskar den tredje forskningsfrågan och fokuserar på att bredda idéer för undervisning i tidig algebra i de lägre årskurserna, genom att integrera kritisk matematikundervisning och socio-ekologiska utmaningar som världen står inför idag (se t.ex. Coles, 2024). Denna del presenteras i kapitel 4, som belyser befintlig forskning som stödjer denna expansion, och kapitel 5, som presenterar de empiriska experimenten. Kapitel 6 beskriver resultaten i relation till den tredje forskningsfrågan; Kapitel 7 lägger fram en avslutande diskussion som sammanfogar trådar och öppnar ytterligare möjligheter för framtiden. För att tydliggöra avhandlingens övergripande berättelse kombinerar manuskriptet en sammanhållande kapiteltext med de artiklar som publicerats under forskningsprocessen; artiklarna inkluderas där de är nödvändiga för att läsaren ska kunna följa och förstå avhandlingsarbetets framskridande.

Teoretisk orientering

Avhandlingens teoretiska orientering vilar på två kompletterande ramverk: Learning Activity Theory (LAT) (Davydov, 2008) och Critical Mathematics Education (CME) (Skovsmose, 1994). I den första delen används LAT (med rötter i Vygotskijs kulturhistoriska perspektiv och Leontievs verksamhetsteori, särskilt utvecklad av Davydov) för att förstå hur yngre elevers algebraiska kunnande kan utvecklas genom gemensamt teoretiskt arbete i problematiserande problemsituationer. Centrala didaktiska principer är problemsituation, lärandemodell och motsättningar, som tillsammans syftar till att rikta elevers uppmärksamhet mot kritiska aspekter av algebraiska strukturer och relationer samt stödja kollektiv reflektion och argumentation (se t.ex., Eriksson et al. 2019).

I avhandlingens senare delar breddas denna didaktiska och kunskapsteoretiska grund genom CME (med utgångspunkt i bland annat Skovsmose), där matematik förstås som icke-neutral och i handling, matematik och matematikundervisning i sitt sammanhang, matematikundervisning för medborgarskap i elevernas livsvärldar, och där undervisningens syfte också kopplas till demokrati, makt och medborgarskap. CME bidrar med begrepp som kritiska situationer, livsvärldar, tänkande och realiserade abstraktioner samt matematikens formaterande kraft (se t.ex., Skovsmose, 1994, 2021, 2023a, 2024). Sammantaget används LAT och CME som teoretiska ramverk för att (1) förstå villkor för elevers deltagande i tidig algebra och (2) ”imaging”, utveckla och analysera AWP som problemsituationer där algebraiska och socio-ekologiska aspekter integreras samtidigt i elevernas arbete.

Metodologisk orientering

Avhandlingens metodologiska orientering präglas av en praktiknära ansats, där problem situationer av olika slag utvecklas i nära samarbete med lärare, lärarstudenter och forskare. I licentiatdelen genomförs två empiriska studier med learning study (se t.ex., Marton & Booth, 1997; Marton et al., 2004) som övergripande forskningsansats. Arbetet organiserades som en cyklisk och kollaborativ process där lärare och forskare gemensamt planerade, genomförde och analyserade forskningslektioner med fokus på att skapa villkor för yngre elevers algebraiska arbete. Datamaterialet utgjordes bland annat av lektionsplaneringar, videoinspelade lektioner, transkriptioner av utvalda sekvenser samt elevintervjuer, och analyserna används för att identifiera kritiska aspekter av elevers generaliserings- och struktureringsarbete samt didaktiska principer som kan stödja detta.

I avhandlingens senare delar användes Skovsmose och Borbas (2004) modell för kritisk matematikdidaktisk forskning för att ”imaging”, pröva med elever och vidareutveckla Algebraic Wicked Problems (AWP) som en ny typ av problemsituationer där tidig algebra integreras med socio-ekologiska frågor. Ett forskarlag med forskare, yrkesverksamma lärare och lärarstudenter etableras för att möjliggöra ett kollaborativt utforskande. Dataproduktionen omfattade exempelvis videoinspelade och transkriberade forskarlags möten, designade AWPs, fältanteckningar från genomföranden av AWPs samt dokumentation av elevers arbete med AWPs. Analysen är huvudsakligen kvalitativ och abduktiv se t.ex., Alvesson & Sköldberg, 2018), med inslag av kvalitativ innehållsanalys (se t.ex., Mayring, 2015) där materialet bearbetas i flera steg för att identifiera återkommande mönster och successivt precisera kännetecken för AWP.

Resultat

Avhandlingens resultat visar sammantaget hur problem sitatuioner kan utformas för att möjliggöra för yngre elevers deltagande i algebraisk aktiviteter, samt hur tidig algebra undervisning kan relateras till medborgarskap i en socioekologisk samtid.

I licentiatdelens första empiriska studie identifieras kritiska aspekter som kan ses som avgörande för att elever i årskurs 2–3 ska kunna uttrycka och argumentera för mönstergeneraliseringar algebraiskt. Resultaten pekar särskilt på betydelsen av att elever får möjlighet att urskilja relationen mellan en figurs position och antalet ingående komponenter, att använda denna relation för att förutsäga en godtycklig (icke given) figur i mönstret samt att uppmärksamma det som är konstant i mönstret. Dessa aspekter synliggör hur elevers sätt att strukturera mönster hänger samman med deras möjligheter att föra resonemang om generella och relationella aspekter.

Den andra licentiatstudien visar vidare att didaktiska principer från LAT – problemsituation, lärandemodell och motsättningar –

kan fungera som redskap för att etablera och upprätthålla ett algebraiskt arbete i klassrummet (se t.ex., Eriksson, 2019). Problemsituationens utformning och lärarens iscensättning framstår som avgörande för att rikta elevers uppmärksamhet mot kritiska aspekter. Lärandemodellerna får en dubbel funktion: de visualiserar centrala relationer och fungerar som gemensamma redskap för kommunikation, argumentation och kollektiv reflektion. Motsättningar – i form av ”snag” eller dilemman – bidrar till att skapa behov av fördjupat utforskande och till att upprätthålla det algebraiska arbetet.

I avhandlingens mittdel visar analysen av två forskningssammanställningar (Cai & Knuth, 2011och Kieran, 2018a) inom tidig algebra hur syften och centrala frågeställningar inom tidig algebra artikuleras i forskningen. I resultaten framträder hur forskningen i hög grad betonar algebra som “powerful” i logisk och psykologisk mening (t.ex. begreppslig förståelse och kognitiv utveckling), medan kulturella och sociologiska tolkningar av algebra som ”powerful” – kopplade till livsvärldar, samhällsfrågor och medborgarskap – framträder mer sparsamt. Detta resultat motiverar ett behov av att utveckla undervisnings- och forskningsansatser som i högre grad integrerar tidig algebra med samhälleligt relevanta frågor.

Det mest framträdande resultatet i del 2 och 3 är utvecklingen och preciseringen av AWP som en typ av problemsituation där algebraiska och socio-ekologiska aspekter ges likvärdig betydelse och där arbetet inte reduceras till att nå ett entydigt ”rätt svar”. Analysen leder till fem karaktärisktiska egenskaper av AWPs: (1) AWPs involverar kulturellt och sociologiskt kraftfulla algebraiska idéer i elevers livsvärldar, (2) AWPs involverar kritiska situationer i elevers livsvärldar; (3) AWPs involverr olika dimensioner av abstraktioner (4) AWPs involverar inbyggda dilemman och värdekonflikter, samt (5) AWPs involverar schematiska, psykologiska och sociologiska redskap som materialiserare. Tillsammans visar resultaten hur AWP kan fungera som en didaktisk bro mellan tidig algebra och socio-ekologiska frågor, och hur undervisning kan utformas så att elever både utvecklar algebraiskt tänkande och ges möjlighet att orientera sig som kritiska medborgare i en komplex och osäker samtid.

Sammanfattningsvis kan AWP förstås som ett analytiskt och didaktiskt ramverk som operationaliserar och sammanför tre forskningsinriktningar, vilka i kapitel 4 identifieras som centrala för att orientera undervisning i tidig algebra avseende såväl innehåll som arbetssätt, samtidigt som aspekter av samtida socioekologiska kriser beaktas. För det första vidareutvecklas Davydovs begrepp om lärandemodeller genom att förena schematiska modeller, tabeller, diagram och konkreta artefakter med socio-kritisk modellering (t.ex. Barbosa, 2006; Coles, 2023b; de Freitas et al., 2022; Solares et al., 2022), där matematiska relationer inte bara visualiseras utan också gör värden, intressen och konflikter i elevernas livsvärld synliga. För det andra bygger AWP på forskning om matematiska modeller i kritiska situationer och matematikens formaterande makt (e.g., Kwon et al., 2021; Skovsmose, 1994; Steffensen & Karceja, 2021), och undersöker hur denna kan kopplas till samspelet mellan olika former av abstraktion, tänkande abstraktoner och reliserade abstraktioner (Skovsmose, 1994), i yngre elevers arbete med kritiska socio-ekologiska situationer. För det tredje bygger AWP:er på forskning om begreppet wicked problems och hur detta kan användas i undervisning (t.ex. Rittel & Webber, 1973; Block et al., 2019; Kynigos, 2024; Lönngren & van Poeck, 2021; Steffensen et al., 2018), men för in denna typ av problem i en tidig algebraisk praktik, där den tidiga algebrans roll också blir en resurs för att utveckla en känslighet för komplexiteten i socioekologiska frågor. Detta möjliggör, vad Haraways (2016) formulerar som att ”stay in the trouble”.

Sammanfattande diskussion

Avhandlingens teoretiska bidrag belyser vad, hur och varför tidig algebra blir betydelsefull i relation till kritiska socio-ekologiska situationer i yngre elevers livsvärldar. Därigenom knyts avhandlingens övergripande ambition ihop: att visa hur tidig algebra kan bidra både till elevers matematiska utveckling och till en begynnande kritisk medborgarförmåga i en samtid präglad av socio-ekologiska utmaningar.

Ett annat huvudbidrag är den komplementära användningen av LAT och CME. Teorierna används ofta i olika riktningar—LAT för att utforska vad Skovsmose och Valero (2008) benämner kraftfulla matematiska idéer i logiska och psykologiska termer, och CME för att förstå matematikens kulturella och sociologiska dimensioner. I en produktiv spänning mellan dem blir det möjligt att förstå ”kraftfulla” idéer både som kognitiva strukturer och som något som formas av och verkar i sociala, politiska, historiska och etiska sammanhang. Kopplade till och konkretiserade i AWPs synliggörs didaktiska möjligheter att utforma undervisning där algebraiska och socio-ekologiska aspekter ges likvärdigt utrymme och förstås som ömsesidigt beroende.

Vidare diskuteras tidig algebra i en socio-ekologisk samtid, formulerad som att kunna algebra i kritiska situationer. I kontrast till tidigare forskning, som främst beskriver tidig algebra som arbete med mönster, strukturer och generaliseringar i matematiska termer, framstår algebraiskt kunnande i AWPs som situationsbundet och praktiknära. Algebraisk aktivitet tar form i sammanhang där social-ekologiska relationer, konsekvenser och värden står på spel. Algebraiskt kunnande förstås därmed inte som en isolerad färdighet, utan som en del av samband mellan matematik i handling, matematikens formaterande kraft och de kriser som präglar elevers livsvärldar. I denna inramning integreras också värdefrågor, till exempel gällande koldioxidutsläpp och dess påverkan, i det algebraiska arbetet.

Resonemanget fördjupas genom att diskutera hur elever i sitt arbete med AWPs relaterar och strukturerar i handling genom att röra sig mellan olika former av abstraktion. I arbete med matematiska representationer, exempelvis tabeller, framträder hur elever kan koppla tänkande abstraktion (Skovsmose, 1994) till det som representationen refererar till och samtidigt uppmärksamma hur representationer kan påverka tolkningar, bedömningar och beslut—det vill säga matematikens formaterande roll (Skovsmose, 1994). Därmed tydliggörs hur tidig algebra kan bidra till både algebraisk förståelse och en begynnande förmåga att kritiskt reflektera över matematikens funktion i samhälleliga frågor.

Avhandlingen avslutas med en framåtblickande diskussion, ”Orienting the gaze toward the future”, där konceptualiseringen av AWP och deras potential att engagera studenter i att samtidigt integrera algebraiska och socioekologiska aspekter ger en värdefull grund för framtida framsteg. Vidare framhålls att de behöver prövas och vidareutvecklas i fler sammanhang. Det vill säga att det finns en önskan att vidareutveckla och pröva AWP:er i fler klassrum och kring fler teman. Avslutningsvis pekas det mot fortsatt samarbete och forskning, bland annat genom transdisciplinära perspektiv och en relationell förståelse av kunskapande — där elever ses som ”själv-i-relation” och kunnande som förståelse av sin position i dynamiska nätverk av sammanhang och betydelser (se t.ex., Abthai, 2022).

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Appendix 1 – Letter to pre-service teachers

Rubrik i brevet: Brinner du precis som jag för att skapa förutsättningar för elevers lärande?

Hej!

Brinner du precis som jag för att lära dig mer om hur vi kan skapa förutsättningar för elevers lärande genom undervisning? Jag heter Jenny Fred och har arbetat som lärare under flera år i Stockholm stad. Numera är jag doktorand och lärarutbildare på MND på Stockholms universitet. Just nu försöker jag få kontakt med lärarstudenter som är nyfikna på att, tillsammans med mig och lärare som undervisar i årkurserna F-3, designa och utveckla problemsituationer som stödjer yngre elevers framväxande algebraiska tänkande. Jag har tidigare deltagit i olika forskningsprojekt där jag tillsammans med lärare och andra forskare, i en kollaborativ och cyklisk process, har utforskat och utvecklat kunskap om hur undervisningen kan skapa förutsättningar för elevers lärande. I de tidigare forskningsprojekten har jag upplevt att just integreringen av olika professioner har varit betydelsefull, då vi har kunnat bidra med olika erfarenheter.

Idén med kommande forskningsprojekt är att tillsammans med lärarstudenter och verksamma lärare designa, pröva och utveckla en eller flera problemsituationer som stödjer yngre elevers framväxande algebraiska tänkande. Vidare är idén att detta ska ske i en cyklisk process som kommer att se ut ungefär så här: 1) Vi designar tillsammans en eller flera problemsituationer; 2) Några av oss prövar problemsituationen/problemsituationerna med en liten grupp elever; 3) Vi analyserar och reviderar problemsituationen/problemsituationerna med utgångspunkt i utprövningen; 4) Några av oss prövar den/de reviderade problemsituationen/problemsituationerna med en ny liten grupp elever; 5) Vi analyserar problemsituationen/problemsituationerna med utgångspunkt i utprövningen.

Det som krävs av dig är att du kan träffas via zoom (tror tyvärr inte vi under rådande omständigheter att vi kommer att kunna ses i verkligheten) 3–4 tillfällen (ca 2h/tillfälle) under vårterminen. Om du önskar (är självklart frivilligt) kan du även få pröva problemsituationen/problemsituationerna med några elever (eller barn som du känner och som är inom åldern för årskurs F-3). Tyvärr kommer det inte finnas någon möjlighet för mig att finansiera ditt deltagande men andra ”vinster” som ditt deltagande kan generera i är erfarenheter och kunskaper som kan vara användbara för dig när det är dags att skriva självständigt arbete. Du kommer även att få ett skriftligt intyg att du har deltagit i ett praktiknära forskningsprojekt, vilket kan fungera som en merit för dig när du ska söka lärarjobb.

Är du nyfiken på att bilda ett forskarlag tillsammans med mig och lärare som undervisar i årkurserna F-3, så svara med vändande mejl. Om du anmäler ditt intresse kommer du att få mer information inom kort.
Jag ser fram emot din intresseanmälan!

Om du har några frågor eller funderingar så tveka inte med att höra av dig!

Hälsningar
Jenny Fred

Om du är nyfiken på att läsa om mina tidigare forskningsprojekt kan du bland annat hitta det här:

Fred, J. (2019). Att etablera och upprätthålla ett algebraiskt arbete i årskurs 2 och 3. Stockholm: Institutionen för matematikämnets och de naturvetenskapliga ämnenas didaktik.

Länk: https://www.diva-portal.org/smash/record.jsf?pid=diva2%3A1375052&dswid=7076

Forskningsprojekt finansierat av Skolforskningsinstitutet: Förmågan att föra och följa algebraiska resonemang - utmaningar för grund- och gymnasieskolan
Länk: https://www.su.se/hsd/forskning/forskningsprojekt/förmågan-att-föra-och-följa-algebraiska-resonemang

Appendix 2 – Letter of consent to in-sevice and pre-service teachers

Information om forskningsprojekt

Under vårterminen 2021 genomförs ett forskningsprojekt där lärarstudenter, verksamma lärare och forskare tillsammans designar, prövar och utvecklar en eller flera problemuppgifter som stödjer yngre elevers framväxande algebraiska tänkande. Ett övergripande syfte med projektet är att utveckla algebraundervisningen för såväl elever i årskurs F-3 som för lärarstudenter som går grundlärarprogrammet med inriktning mot årskurs F-3.

Hur går studien till?
Inom projektet kommer det att genomföras bild/ljudupptagning av forskarlagets möten och av utprövandet av problemuppgifter med elever. Vidare kan lärarstudenternas och lärarnas texter/anteckningar, planeringar, utkast på gemensamt designade problemuppgifter, elevers lösningar att komma att samlas in. Delar av materialet kan komma att användas i utbildningssyfte för lärare, lärarstudenter, i föreläsningar, seminarier eller i forskningssammanhang. Vid sådana tillfällen kommer filmklipp aldrig distribueras vidare utan endast att visas. Forskningsprojektet förväntas resultera i vetenskapliga publikationer och presentationer. Skolan, lärare, lärarstudenter och de elever som väljer att vara med i forskningsprojektet kommer att anonymiseras i material som publiceras.

Vad händer med de uppgifter som samlas in?

Bild/ljudupptagningar klassas som personuppgifter och då finns det vissa regler som forskare behöver följa. Ansvarig för personuppgifterna är Stockholms universitet. Allt material som vi samlar in kommer att behandlas i enlighet med god forskningssed och dataskyddsförordningen vilket bl. a. innebär att det kommer att förvaras och behandlas så att inte obehöriga kan ta del av det. Bild/ljudupptagningarna kommer att lagras vid en för Stockholms universitet gemensam lagringsyta där bara specifika användare kan nå specifik data. Efter 10 år kommer materialet att förstöras.

Bild/ljudupptagningarna kommer inte att få användas i syftet att bedöma enskilda elevers eller lärarstudenters prestationer.

Deltagande forskare/lärarstudenter/lärare i genomförandet av datainsamlingen är: xxxxx.
Vetenskapligt ansvarig för projektet är, Hendrik van Steenbrugge, Institutionen för matematikämnets och naturvetenskapsämnenas didaktik, Stockholms universitet (mail: hendrik.vansteenbrugge@mnd.su.se). Deltagande doktorand/lärarstudenter/lärare samtycker genom underskrift nedan till att:

- Det insamlade materialet får endast användas i enlighet med syftet ovan och med godkännande från Hendrik van Steenbrugge och Jenny Fred.

- Bild/ljudupptagningar kommer att lagras på ett sådant sätt att inga obehöriga kan få tillgång till dem.

- Vid redovisning och diskussioner av studien ska alla deltagande elever, lärarstudenter och lärare behandlas konfidentiellt.

- Insamlat material kommer inte att användas för att bedöma enskilda elevers eller lärarstudenters prestationer. Analyser kommer inte användas på ett sådant sätt att enskilda elever eller lärarstudenter missgynnas av medverkan eller val att inte medverka (t.ex. i form av sämre betyg).

• Deltagande elever har informerats muntligt och skriftligt om studien samt lämnat skriftligt samtycke om att delta.

• Elevernas samtycken förvaras inlåsta och skilt från bild/ljud-upptagningarna.

• Elevernas samtycken förvaras inlåsta och skilt från bild/ljud-upptagningarna.

Samtycke till att delta i studien
Jag har fått muntlig och skriftlig information om vad det innebär att delta i forskningsprojektet och jag har haft möjlighet att ställa frågor. Jag får behålla en kopia av den skriftliga informationen.

☐ Jag har läst ovanstående information om forskningsprojektet och samtycker enligt ovan.

Namnunderskrift: ____

Namnförtydligande: ____

Ort & datum: ____

Appendix 3 – Letter of consent to guardians and students

The wicked world of algebra:

Ett praktikutvecklande forskningsprojekt i åk F-3

Ert barns skola kommer under höst- och vårterminen 2021/2022 att delta i ett forskningsprojekt med fokus på att utveckla algebraundervisningen för elever i yngre åldrar. Projektet kommer att genomföras i årskurserna F-3 och ert barns klass har blivit utvald av lärarna som undervisar i klassen. Eleverna kommer att få muntlig information om projektet under skoltid och då också få möjlighet att ställa frågor om det. I det här dokumentet får du som vårdnadshavare information om vad det innebär att låta ditt barn delta i forskningsprojektet.

Vad är det för projekt och varför ska du/ditt barn delta?

Syftet med forskningsprojektet är att genom en iterativ process designa, pröva och utveckla problemuppgifter som stödjer yngre elevers framväxande algebraiska tänkande och deras agens, det vill säga handlingskapacitet. Det är alltså problemuppgifterna och lärarens undervisning i klassen som utgör studiens huvudfokus.

Datainsamling och datahantering

Ert barn kommer att delta i en, två eller tre forskningslektioner. Forskningslektionerna kommer att ersätta motsvarande ordinarie lektioner i matematik.

Forskningslektionerna kommer dokumenteras med fotografering. Ditt barns ansikte kommer inte att synas på dessa foton. Även skriftligt material från dessa kommer att samlas in för att vi efteråt ska kunna analysera innehållet. Det insamlade materialet kommer användas inom projektet med fokus riktat på undervisningens innehåll och inte på enskilda individer. Vi kommer inte att upprätta en lista över namnen på de elever som deltar i studien.

Vad händer med mina/ditt barns uppgifter i studien?

Vetenskapligt ansvarig för projektet, Hendrik Van Steenbrugge, Institutionen för matematikämnets och naturvetenskapsämnenas didaktik, Stockholms universitet (mail: hendrik.vansteenbrugge@mnd.su.se), ansvarar för att förvaringen av datamaterialet sker enligt de forskningsetiska principer som gäller för all forskning. Elevernas svar på skriftliga och muntliga frågor samt de dokumenterade lektionerna kommer att förvaras på ett sådant sätt att inte obehöriga kan ta del av dem. Datamaterialet kommer att sparas enligt Arkivlagen och Riksarkivets allmänna föreskrifter.

Mer information om de forskningsetiska principerna finns på denna länk:
http://www.codex.vr.se/manniska2.shtml

När vi presenterar våra resultat kommer vi att använda fingerade namn. Detta innebär att ingen utanför forskargruppen kommer att veta vad en enskild elev har sagt eller gjort. Studiens resultat kommer att tillgängliggöras i form av till exempel tidskriftsartiklar, böcker och konferenspresentationer och delar av materialet kan komma att användas i utbildningssyfte för lärare, i föreläsningar eller seminarier. I båda fall kommer vi att använda fingerade namn.

Personuppgiftshantering
Det som enligt den nya lagen (EU:s dataskyddsförordning 2016/679, GDPR) räknas som personuppgifter i denna studie är fotografier där det går att identifiera en enskild person, i det här fallet en elev. Personuppgiftshanteringen sker med stöd av
givet samtycke.

Ert barns deltagande i studien är frivilligt och ni som vårdnadshavare har rätt att
- när som helst under studien återkalla ett givet samtycke
- begära tillgång till ert barns personuppgifter
- få personuppgifterna rättade
- få personuppgifterna raderade
- få behandlingen av personuppgifterna begränsad
- inge klagomål till datainspektionen

Personuppgiftsansvarig är Stockholms universitet.

Vi hoppas att ni som vårdnadshavare, liksom vi, ser värdet av denna forskning och samtycker till att ert barn deltar i projektet. ***Ditt barns deltagande i studien är frivilligt och du kan när som helst avbryta ditt barns deltagande utan närmare motivering. Ditt barn har också rätt att när som helst meddela att det sitt deltagande önskar avbryta.*** Om ni önskar avbryta eller på annat sätt ändra givet samtycke går det bra att kontakta forskningsledaren eller någon annan av de medverkande forskarna/lärarna (se nedan). Det går också bra att kontakta universitetets dataskyddsombud, Benita [illegible] mailto:dso@su.se, tel. 08-16 41 91. Om du har klagomål på vår behandling av dina personuppgifter har du rätt att inge klagomål till tillsynsmyndigheten Datainspektionen på datainspektionen@datainspektionen.se.

På nästa sida finns en samtyckesblankett. Vänligen besvara denna samtyckesblankett och returnera den till ditt barns lärare senast xxx.

Vänliga hälsningar,
Jenny Fred

Övriga medverkande forskare i projektet under läsåret 2021/2022 är:
Carina Brink, Emelie Rosendahl, Eva [illegible], Hendrik van [illegible], Jennie Rundqvist, Mikael Åström, Paola Valero och Pär [illegible]

Har ni några frågor kontakta Jenny Fred, mail: jenny.fred@mnd.su.se

Samtycke till att delta i studien

Mitt barn har fått muntlig information och har haft möjlighet att ställa frågor om studien xxx. Vi har tagit del av den skriftliga informationen om studien och vi som vårdnadshavare får behålla den skriftliga informationen.

I och med de nya personuppgiftsreglerna (GDPR) måste samtliga vårdnadshavare samtycka till att ett barn under 15 år deltar i ett forskningsprojekt.

☐ Vi samtycker till att vårt barn (namn)______________________________________ deltar i studien "The wicked [illegible] of algebra: Ett praktikutvecklande forskningsprojekt i åk F-3" samt att uppgifter om vårt barn behandlas på det sätt som beskrivs i ovan.

☐ Nej, jag vill inte delta/att mitt barn deltar i studien.

Vårdnadshavare 1:

Namnunderskrift:

Namnförtydligande:

Ort & datum: ____

Vårdnadshavare 2:

Namnunderskrift:

Namnförtydligande: ____

Ort & datum:

Elev:

Namnunderskrift: ____

Namnförtydligande:

Ort & datum:

Appendix 4 – Certificate for participation

Stockholms universitet

INTYG

xxxx

har deltagit i

The Wicked World of Algebra - A "practice-based" research project in grade F-3

xxxx har under vt 2021 varit en av deltagarna i ett forskarlag där vi i en iterativ och kollaborativ process tillsammans har utforskat vad algebraiska "wicked problems" i en F-3 kontext kan vara. Emelie har varit en stor tillgång i forskarlaget och har fört arbetet framåt genom sin förmåga att komma med ideér, reflektera över sina och andras ideér samt ställa kritiska och utvecklande frågor.

Jenny Fred
Fil.lic. i matematikämnets didaktik
Institutionen för matematikämnets och naturvetenskapsämnenas didaktik
jenny.fred@mnd.su.se

This thesis investigates how socio-ecological considerations can be integrated into the teaching and learning of early algebra in primary school. Drawing on Learning Activity Theory and Critical Mathematics Education, the study examines how students' algebraic activity can be organized.

The findings conceptualize a type of problem situation termed Algebraic Wicked Problems (AWPs), characterized by the simultaneous integration of algebraic and socio-ecological aspects. The analysis identifies five key features of AWPs and shows how these problems can support younger students' participation in algebraic activities while connecting early algebra to questions of citizenship in a socio-ecological world. Overall, the thesis contributes conceptual tools for designing and analyzing early algebra teaching at the intersection of learning, teaching, and societal conditions, and it demonstrates how early algebra may be oriented toward both mathematical development and acting/knowing in critical situations.

ISBN 978-91-8107-622-6

Department of Teaching and Learning

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