Stockholm University

Representations of Toposes

Errol Yuksel

Doctoral Thesis in Mathematics at Stockholm University, Sweden 2026

Representations of Toposes

Errol Yuksel

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics at Stockholm University to be publicly defended on Monday 15 June 2026 at 13.00 in Lärosal 22, Hus 4, Albano, Albanovägen 12 and online, public link is available at the department website.

Abstract

This thesis is based on three papers which focus on representations of Grothendieck toposes as localic groupoids, localic stacks, and categories of points equipped with convergence data.

In Paper I, we analyse various localic covering theorems and localic groupoid representations for toposes given in the literature. We note that each of these arises from a minimal, concrete object: a locale equipped with a suitable '*amorphous*' sheaf. With this definition in hand, we abstract the standard recipe for covering theorems, describe and compare the amorphous sheaves associated to standard covering constructions from the literature, and give a general logical characterisation of amorphous objects.

In Paper II, we show that the 2-category of locales is dense in that of toposes, and that toposes can faithfully be represented as stacks over locales. This is achieved by the combination of an original technical result about the 2-localisation induced by a 2-site, and the key observation that open surjections are of lax descent type in toposes.

In Paper III, we build on the recent development that toposes with enough points can be represented as categories equipped with ultraconvergence data, using it to characterise the surjection–embedding and hyperconnected–localic factorisation systems in terms of points. We introduce analogous classes of functors between abstract categories equipped with ultraconvergence data; provide a new characterisation of separating families of points of a topos; and apply it to show preservation and reflection results between these classes of functors and geometric morphisms.

Keywords: *topos, locale, stacks.*

Stockholm 2026
http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-254626

ISBN 978-91-8107-660-8
ISBN 978-91-8107-661-5

Department of Mathematics

Stockholm University, 106 91 Stockholm

REPRESENTATIONS OF TOPOSES

Errol Yuksel

Stockholm University

Representations of Toposes

Errol Yuksel

ISBN print 978-91-8107-660-8
ISBN PDF 978-91-8107-661-5

Printed in Sweden by Universitetsservice US-AB, Stockholm 2026

Abstract

This thesis is based on three papers which focus on representations of Grothendieck toposes as localic groupoids, localic stacks, and categories of points equipped with convergence data.

In Paper I, we analyse various localic covering theorems and localic groupoid representations for toposes given in the literature. We note that each of these arises from a minimal, concrete object: a locale equipped with a suitable '*amorphous*' sheaf. With this definition in hand, we abstract the standard recipe for covering theorems, describe and compare the amorphous sheaves associated to standard covering constructions from the literature, and give a general logical characterisation of amorphous objects.

In Paper II, we show that the 2-category of locales is dense in that of toposes, and that toposes can faithfully be represented as stacks over locales. This is achieved by the combination of an original technical result about the 2-localisation induced by a 2-site, and the key observation that open surjections are of lax descent type in toposes.

In Paper III, we build on the recent development that toposes with enough points can be represented as categories equipped with ultraconvergence data, using it to characterise the surjection–embedding and hyperconnected–localic factorisation systems in terms of points. We introduce analogous classes of functors between abstract categories equipped with ultraconvergence data; provide a new characterisation of separating families of points of a topos; and apply it to show preservation and reflection results between these classes of functors and geometric morphisms.

Sammanfattning på svenska

Denna avhandling består av tre artiklar om representationer av Grothendieck-toposer som lokaliska gruppoider, lokaliska stackar och kategorier av punkter utrustade med konvergensdata.

I Artikel I analyserar vi olika lokaliska övertäckningssatser och lokaliska gruppoidrepresentationer för toposer som förekommer i litteraturen. Vi noterar att de härstammar från ett minimalt, konkret objekt: en lokal utrustad med ett lämpligt '*amorft*' kärve. Med denna definition till hands abstraherar vi standardreceptet för övertäckningssatser, beskriver och jämför de amorfa kärvar som är associerade med standardövertäckningskonstruktioner från litteraturen, samt ger en allmän logisk karakterisering av amorfa objekt.

I Artikel II visar vi att 2-kategorin av lokaler är tät i den av toposer, och att toposer troget kan representeras som stackar över lokaler. Detta uppnås genom en kombination av ett originaltekniskt resultat om 2-lokalisationen som induceras av en 2-plats, och den viktiga iakttagelsen att öppna surjektioner är av lax nedstigningstyp i toposer.

I Artikel III bygger vi vidare på den senaste utvecklingen att toposer med tillräckligt många punkter kan representeras som kategorier utrustade med ultrakonvergensdata, och använder detta för att karakterisera surjektion–inbäddning och hypersammanhängande–lokalisk faktoriseringssystem i termer av punkter. Vi introducerar analoga klasser av funktorer mellan abstrakta kategorier utrustade med ultrakonvergensdata, ger en ny karakterisering av separerande familjer av punkter i en topos, och tillämpar detta för att visa bevarande- och reflektionsresultat mellan dessa klasser av funktorer och geometriska morfismer.

List of Papers

The following papers, numbered I–III, are included in this thesis.

Paper I: **Toposes vs Localic Groupoids: a unified treatment of covering theorems.**
Ivan Di Liberti, Peter LeFanu Lumsdaine, and Errol Yuksel. *Preprint.* 2026.

Paper II: **Toposes as localic stacks.**
Errol Yuksel. *Preprint.* 2026.

Paper III: **Geometric morphisms of virtual ultracategories.**
Gabriel Saadia and Errol Yuksel. *Preprint.* 2026.

Papers I and III are the results of collaborative work in which all parties contributed equally.

Acknowledgements

Most of all, I would like to thank my supervisor Peter LeFanu Lumsdaine, whose support and encouragement were of crucial help during the writing of this thesis. I am especially grateful for the perfect balance between freedom and guidance he managed to strike; enough freedom that I felt comfortable changing topic and enough guidance that I never felt lost.

I would also like to thank Ivan Di Liberti, both for properly introducing me to topos theory during his stay as a postdoc in Stockholm, and for always being so generous with his time during our mathematical discussions.

My thanks also go to the whole of the Stockholm Logic group, past and present members alike, for stimulating seminars, very enjoyable logic dinners, and overall for fostering the social side of mathematical research.

More broadly, I would like to acknowledge how engaging a work environment the Mathematics Department has been, due in no small part to my fellow PhD students, among which I would particularly thank Benedetta and Gabriel for the interesting conversations we shared.

Lastly, but by no means least, I want to express how grateful I am to have found such great friends as I did in Anton, Dan, Felix, Haily, and Silver, and thank them for the great times we shared together.

Contents

General background in topos theory

During his search for ‘Weil cohomologies’ with the properties needed to prove the Weil conjectures, Grothendieck came upon a notion which merged aspects of usual sheaf cohomology and of Galois cohomology, the so-called *étale cohomology*. Its definition required a generalisation of the concept of sheaves on a space to sheaves on a *site*, a category equipped with a notion of *covering* abstracting the open covers of a space used to define a sheaf. In a sense, sites categorify exactly the aspects of topological spaces needed to define the concept of sheaf on a site. Unlike with topological spaces, two non-equivalent sites can give rise to the same category of sheaves. This suggested to Grothendieck that the categories of sheaves on sites, and not the sites themselves, were the right objects to study — not unlike the relationship between presentations of groups and groups.

The notion of *topos*, specifically speaking of what is now called Grothendieck topos, arose in this way as an abstraction of the categories of sheaves on a site. The initial study of sites, sheaves and toposes was carried out in [SGA4.1]. As it was designed for, the resulting theory proved of extreme use for algebraic geometry in the crucial role it played in Deligne’s proof of the Weil conjectures. One of the main guiding directions, if not the main one, of topos theory is the analogy between sheaves on a site and sheaves on a topological space, which suggest that toposes are a notion of generalised, categorified space. Grothendieck even wrote [SGA4.1, p. 301] that the name ‘topos’ was precisely chosen to suggest that the actual object of topology is the study of toposes and not only of topological spaces.

Already in [SGA4.1], the term ‘topos classifiant’ is used for the topos of G-sets of a given group G, since it represents G-torsors. Monique Hakim, a student of Grothendieck, introduced more refined examples of such objects in her PhD thesis [Hak72], such as the toposes representing the theory of rings and local rings. These are the oldest examples of what are now called *classifying toposes*, but at that stage the scope of structures which can thus be represented by toposes is quite restricted. The first people to consider that toposes could classify a wide class of logical theories seem to have been Gonzalo Reyes and his student Jean Dionne. Together they construct the syntactic category of a coherent theory, endow it with a logical topology, and show that the topos of sheaves on the resulting site contains a generic model of the theory.

This work was then taken up by many authors such as Johnstone, Joyal, Lawvere, and Wraith. The first book account of this relationship between Grothendieck toposes and logical theories was given in Makkai–Reyes [MR77], where they also generalised from usual finitary first-order logic (also known as coherent logic) to its infinitary variant that became *geometric logic*. At the same time, other approaches were being devised by other authors, such as a ‘step-by-step’ construction of classifying toposes stemming from the independent work of Joyal, Tierney, and Bénabou; this method was then recast in purely 2-categorical terms by Cole and Johnstone.

This logical point of view, where every geometric theory admits a classifying topos and every topos classifies some non-canonical theory, can be understood as a way to represent toposes. This is entirely analogous to the way sites represent toposes: every site induces a topos and every topos arises from some non-canonical site. This parallelism between geometric theories and sites is one of the key insights behind the technique of toposes as ‘bridge’ developed by Caramello [Car18]. The idea is that while sites and theories can both be used as representing objects for toposes, they remain quite different objects. It is more likely that examples from topology or algebraic geometry can be expressed in terms of the former than the latter, and conversely with examples from logic or model theory. Nonetheless, the topos of sheaves over such a site could agree with the classifying topos of such a theory, and the common topos would then serve as a bridge between these distant examples.

There exists many more ways to represent and think of toposes than sites and theories[1], which is a central reason for the richness and depth of topos theory. In this thesis, we explore three different such representations, described in more detail in the three sections below. The first, toposes as localic groupoids, is the more established of the three and goes back to work of Joyal–Tierney [JT84] but was then expanded upon by a wealth of authors. Our contribution is a unifying one and aims to compare and contrast the different reconstruction theorems of toposes in terms of localic groupoids. The second, toposes as localic stacks, is novel and aims to portray toposes not as categories of sheaves but as higher sheaves (stacks) over a category of spaces (locales). This is a more 2-categorical and conceptual way to think about toposes. The third, toposes as ultracategories, goes back to Makkai [MR77] for coherent toposes but was recently expanded to toposes with enough points in three independent works [Saa25; Ham25; GMT26]. The idea is to see toposes as categories of points equipped with (ultra)convergence data. We use this point of view to characterise important classes of geometric morphisms solely in terms of points.

[1] Johnstone lists thirteen in the preface of [Joh02].

1 Toposes as spatial groupoids

1.1 Locales

Toposes are categorified spaces insofar as they abstract categories of sheaves over topological spaces. One can play this game one dimension lower, and consider objects which abstract the opens of a space.

Definition. A *frame* is a partially ordered set A with all joins $\bigvee$ and finite meets $\wedge$, which satisfies the infinite distributivity property:

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

for each $a, (b_i)_{i \in I}$ in A. A map of frames is one that preserves all joins and finite meets, and the resulting category is denoted by Frm.

In other words, frames are lattices which mimic the properties satisfied by the opens of a space. The poset $\mathcal{O}(X)$ of opens of any topological space X is a frame by design. Moreover, if $f : X \to Y$ is a continuous map then its inverse image $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a map of frames. This means that sending a space to its frame of opens defines a functor

$$\mathcal{O} : \mathsf{TopSp} \longrightarrow \mathsf{Frm}^{\mathrm{op}}$$

which flips the direction of arrows. This motivates studying $\mathsf{Frm}^{\mathrm{op}}$, the opposite category of Frm, rather than Frm itself so as to ensure that morphisms go in the geometric direction. Note that frames in the image of $\mathcal{O}$ are all subframes of a powerset $\mathcal{P}(X)$ by definition, and that there is no clear reason to assume that this would be true of all frames — more on this soon.

The category of *locales* is defined as the formal opposite of the category of frames, $\mathsf{Loc} := \mathsf{Frm}^{\mathrm{op}}$. One can think of it as the first step from topological spaces to toposes. Instead of directly considering abstract categories of sheaves, one starts with abstract posets of opens. Locales and frames can also be understood as the portion of topological spaces relevant to sheaf theory. Indeed, the definition of sheaf depends only on the frame of opens of a space and not on the points. Indeed, one can define sheaves over any locale, not just those in the image of $\mathcal{O}$.

Definition. Let A be a locale. A sheaf over A is a functor $F : A^{\mathrm{op}} \longrightarrow \mathrm{Set}$ such that

$$F(a) \longrightarrow \prod_{i\in I} F(b_i) \rightrightarrows \prod_{i,j\in I} F(b_i \wedge b_j)$$

is an equaliser diagram for each $a \in A$ and family $(b_i)_{i\in I}$ of A which covers a, in the sense that $a = \bigvee_{i\in I} b_i$.

Just as for topological spaces, the category of sheaves over a locale is a Grothendieck topos, but, in contrast to spaces, the resulting functor $\mathrm{Sh} : \mathsf{Loc} \longrightarrow \mathsf{Topos}$ is much better behaved than the one defined on TopSp. Indeed, without going into the details or proofs, this functor exhibits the category of locales as a reflective subcategory of that of toposes. This means that Sh is fully faithful and admits a left adjoint.

$$\mathsf{Loc} \underset{\mathrm{Sh}}{\overset{\perp}{\hookrightarrow}} \mathsf{Topos}$$

The main contrast between locale theory and the theory of topological spaces is the role of points; locales are sometimes called pointfree spaces (or even pointless spaces [Joh83]). In this context, a point of a locale is just a map from the terminal locale to it. As their alternative names suggest, locales do not necessarily admit points. Of course many do, for example any locale of the form $\mathcal{O}(X)$ for a space X. Notable instances of this pointfree phenomenon, highly relevant for the considerations of Paper I, are the locales of *enumerations.*

If S is any finite set, the set of functions $S^{\mathbb{N}}$ can be seen as a topological space $X_{\mathbb{N}\longrightarrow S}$ with the product topology, and we can look at the subspace $X_{\mathbb{N}\twoheadrightarrow S}$ of surjective functions of that space. These are topological spaces and can thus also be seen as frames, in which case the inclusion

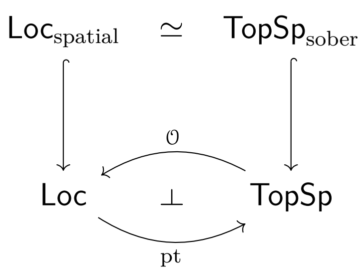
$$X_{\mathbb{N}\twoheadrightarrow S} \hookrightarrow X_{\mathbb{N}\longrightarrow S}$$

becomes a frame quotient

$$\mathcal{O}(X_{\mathbb{N}\twoheadrightarrow S}) \twoheadrightarrow \mathcal{O}(X_{\mathbb{N}\rightarrow S}).$$

We assumed S was finite to define the subspace $X_{\mathbb{N}\twoheadrightarrow S}$ by its points, but its description as a quotient frame works just as well without this assumption. In other words, for any set S, there exists a locale $L_{\mathbb{N}\twoheadrightarrow S}$ of enumerations of S even for infinite S. Points of that locale correspond to surjections $\mathbb{N} \longrightarrow S$, and only exist when S is finite classically.

The elements of a frame L induce a topology on the points of the corresponding locale. Hence, every locale L gives rise to a topological space $\mathrm{pt}(L)$. This defines a functor $\mathrm{pt} : \mathsf{Loc} \longrightarrow \mathsf{TopSp}$ which is a near inverse to $\mathcal{O} : \mathsf{TopSp} \longrightarrow \mathsf{Loc}$, meaning that they form an idempotent adjunction between the categories of locales and topological spaces. Being idempotent means that the adjunction becomes an equivalence when restricted to the respective essential images of $\mathcal{O}$ and pt.



A locale in the image of $\mathcal{O}$ is said to be *spatial* (or to *have enough points*). A space in the image of pt is called *sober*. The intuition behind sober spaces is that they have enough opens to separate their points and that they have all the points that their frames of opens induce. Note that this is the same concept as in algebraic geometry, except expressed in terms of opens instead of irreducible closed sets having generic points.

1.2 Localic groupoids

The transition from spaces to locales gets us closer to toposes, but not all the way. In [JT84], Joyal and Tierney show that the difference between a general topos and sheaves over a locale lies in the possible action by a localic groupoid. That is, a groupoid internal to the category of locales; concretely, a locale of objects G_0 and a locale of morphisms G_1 together with maps

$$G_1 \times_{G_0} G_1 \xrightarrow{\;m\;} G_1 \circlearrowleft \quad \begin{matrix} \xrightarrow{\;s\;} \\ \xleftarrow{\;i\;} \\ \xrightarrow{\;t\;} \end{matrix} \quad G_0$$

satisfying the usual groupoid identities. The resulting category of these objects is denoted by $\mathsf{Gpd}(\mathsf{Loc})$.

Definition. An *equivariant sheaf* over a localic groupoid G is an internal presheaf E over G such that $E \longrightarrow G_0$ is *étale*. Equivalently, a sheaf over G_0 with a compatible G_1-action.

$$\begin{array}{ccccc} t^*E & \xrightarrow{\;e\;} & s^*E & \qquad & E \\ & \searrow \quad \swarrow & & & \big\downarrow \text{étale} \\ & G_1 & & & G_0 \end{array}$$

For example, if L is a locale seen as a (categorically) discrete localic groupoid, then an equivariant sheaf on it is simply a sheaf on L (in the usual meaning). More interestingly, in the topologically discrete case, if G is an actual groupoid (internal to Set that is), then an equivariant sheaf on G is simply a presheaf $G^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

Note that, somewhat counter-intuitively given their names, toposes of presheaves over categories or groupoids are almost as different as one can be from toposes of sheaves over locales. In localic groupoids one can therefore hope to find a middle ground to bridge the gap between these two kinds of toposes, just as one finds in sites.

As one might expect, the category of equivariant sheaves on G forms a topos denoted by $\mathsf{B}G$, and this assignment defines a functor

$$\mathsf{B} : \mathsf{Gpd}(\mathsf{Loc}) \longrightarrow \mathsf{Topos}.$$

Joyal and Tierney showed that this functor is essentially surjective.

Theorem ([JT84])**.** For each topos $\mathcal{E}$, there exists a localic groupoid G such that $\mathsf{B}(G) \simeq \mathcal{E}$.

As mentioned above, this theorem clarified the gap between topological spaces and toposes: one first has to move on to locales, a mild generalisation which puts the opens in the forefront and forgets the points, and then if one adds the data of a groupoidal action the resulting objects can be used to represent all toposes.

Just as with sites, there is no unique (small) localic groupoid representing a given topos. As a result, even though the construction of Joyal and Tierney is quite explicit, one could imagine different ones. Indeed, variations on this theme have since been pursued by many authors (Johnstone [Joh85], Moerdijk [Moe88], Butz–Moerdijk [BM98], Awodey–Forssell [AF13], Wrigley [Wri23; Wri25]), providing a range of constructions of a localic groupoid presentation for a given topos. As already noted in [Joh02, C5.2.8], while all these constructions look different up close, if one takes a step back and analyses the general proof structure a pattern emerges.

1.3 Summary of Paper I

In Paper I, we abstract the pattern shared by all constructions mentioned above and are thus able to distil down these constructions to their core data. These data are localic covers of the object classifier $\mathbf{Set}[\mathbb{O}]$, which can each be represented as a small and quite concrete object: a locale equipped with a suitable sheaf, which we call an *amorphous sheaf*.

We analyse multiple constructions from the literature in terms of their amorphous sheaves, and use this framework to compare them. We also assess the functoriality of these constructions, something often overlooked in the literature and more amenable in our set-up.

Due to their tangible nature, amorphous sheaves can readily be studied from a logical point of view and we provide a logical recognition criterion for amorphous sheaves. This relies on the theory of classifying toposes for first-order theories of Butz–Johnstone [BJ98]. We then use this criterion to extract a seemingly novel covering construction from Kripke's original completeness theorem for intuitionistic predicate logic.

2 Toposes as geometric stacks

In Paper II, we take a different approach than in Paper I and represent toposes through lower-dimensional localic data as canonically as we can. Ideally, one would like to represent a topos in terms of intrinsic data, such as points. However this can fail catastrophically: some toposes have no points at all. Even if one accounts for this and considers only toposes with enough points, the mere 1-category of points of a topos $\mathcal{E}$ is rarely enough to recover $\mathcal{E}$. There are two main ways one can go about this, either one endows the category of points with enough extra structure to recover the topos, or one considers more general points. We will discuss the former in more detail when we come to Paper III. In Paper II, we take the latter approach: we enlarge our notion of point to different bases and show that a topos is canonically represented by this broader category of points.

2.1 Localic points

As with locales, the points of a topos $\mathcal{E}$ are simply the maps of toposes from the terminal topos to it. The terminal topos being $\mathrm{Set} = \mathrm{Sh}(1)$, points of $\mathcal{E}$ are geometric morphisms $\mathrm{Set} \longrightarrow \mathcal{E}$. From a logical point of view, when $\mathcal{E}$ is the classifying topos $\mathrm{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$, points of $\mathcal{E}$ are exactly the (Set-based) models of $\mathbb{T}$. This is because the inverse image of a geometric morphisms preserves logical connectives and entailments, and hence sends the generic model $\mathcal{M}_{\mathbb{T}}$ of $\mathrm{Set}[\mathbb{T}]$ to a model in Set.

We need to consider more general points. The topos Set of sets is terminal topos because we are implicitly considering Set-based toposes, but we can change the base: topos theory is *relative*. This means that working in the slice $\mathsf{Topos}_{/\mathcal{S}}$ amounts to considering $\mathcal{S}$-based toposes (think internal to $\mathcal{S}$).

In that context, points are then geometric morphisms out of $\mathcal{S}$ over $\mathcal{S}$, i.e., sections. Of course, we can change the base of any (Set-based) topos $\mathcal{E}$ to a base $\mathcal{S}$ by pulling back to $\mathcal{S}$, which results in the product topos $\mathcal{E} \times \mathcal{S}$. An $\mathcal{S}$-point of $\mathcal{E} \times \mathcal{S}$ is then a map $\mathcal{S} \longrightarrow \mathcal{E} \times \mathcal{S}$ over $\mathcal{S}$, that is, a geometric morphism $\mathcal{S} \longrightarrow \mathcal{E}$.

$$\begin{array}{ccc} \mathcal{S} & \dashrightarrow & \mathcal{E} \times \mathcal{S} \\ & \diagdown\!\!\diagdown \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

Hence, considering points in more bases than just Set amounts to allowing geometric morphisms $\mathcal{S} \longrightarrow \mathcal{E}$ for some toposes $\mathcal{S}$. By the Yoneda lemma, if we consider points in all possible base extensions of Set, then we are guaranteed a representation of $\mathcal{E}$, but one that is somewhat tautological. We could also restrict to the points $\mathcal{E}/A \longrightarrow \mathcal{E}$ in the bases $\mathcal{E}/A$ for $A \in \mathcal{E}$, which amounts to trying to reconstruct $\mathcal{E}$ from the étale slice $\mathsf{Topos}^{\text{etale}}_{/\mathcal{E}}$. This works too but the base extensions depends on $\mathcal{E}$, which is not exactly what we are after.

In Paper II, we propose allowing for *localic* base extensions, toposes of the form $\mathrm{Sh}(X)$ with X a locale, but no more. This means considering geometric morphisms of the form $\mathrm{Sh}(X) \longrightarrow \mathsf{Topos}$ with X any locale instead of only the terminal locale. In principle this avoids the issue of point-free toposes, as some locales are also point-free, and also excludes points of the form $\mathcal{E}/A \longrightarrow \mathcal{E}$ as those are rarely localic if $\mathcal{E}$ is not itself. In short, we aim to recover a topos $\mathcal{E}$ from its *localic points*.

2.2 Descent and stacks

For locales X, the categories $\mathsf{Topos}(\mathrm{Sh}(X), \mathcal{E})$ of X-points of $\mathcal{E}$ naturally assemble into a pseudo-functor $\mathsf{Loc}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$. A key ingredient in the main proof of Paper II is that this functor is a *stack* for a certain topology on Loc. Stacks are categorified sheaves: whereas sheaves are valued in Set, stacks are valued in Cat. Stacks of groupoids, those valued in in Gpd, are more common in the literature and are often the default meaning of the word stack, but here we required the non-invertible kind of stack. Recall that if $(f_i : x_i \longrightarrow y)_{i \in I}$ is a covering in a category $\mathcal{C}$ and $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a presheaf on $\mathcal{C}$, then F satisfies the sheaf condition with respect to that covering if

$$F(y) \xrightarrow{(f_i^*)} \prod_{i \in I} F(x_i) \underset{t^*}{\overset{s^*}{\rightrightarrows}} \prod_{i,j \in I} F(x_i \times_y x_j)$$

is an equaliser diagram, where s sends $(a_i)_{i \in I} \in \prod_{i \in I} F(x_i)$ to the family $(s_{ij}^* a_i)_{i,j \in I}$ of $\prod_{i,j \in I} F(x_i \times_y x_j)$, and similarly for t, with s_{ij}, t_{ij} the pullback projections.

$$\begin{array}{ccc} x_i \times_y x_j & \xrightarrow{t_{ij}} & x_j \\ {\scriptstyle s_{ij}} \downarrow & \lrcorner & \downarrow {\scriptstyle f_j} \\ x_i & \xrightarrow[f_i]{} & y \end{array}$$

The limit consists of the families $(a_i)_{i \in I} \in \prod_{i \in I} F(x_i)$ for which $s_{ij}^* a_i = t_{ij}^* a_j$ in $F(x_i \times_y x_j)$ for each $i, j \in I$. If the covering is thought of as a family of open subsets whose union covers, then this condition on F says exactly that the data of an element of F over the large subset, y here, is equivalent to the data of objects over the smaller subsets, the x_i's, which 'match together' in that they agree on the intersections. One could imagine extending the diagram above to the right ad infinitum and obtain a simplicial diagram

$$\prod_{i \in I} F(x_i) \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \prod_{i,j \in I} F(x_i \times_y x_j) \begin{smallmatrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{smallmatrix} \prod_{i,j,k \in I} F(x_i \times_y x_j \times_y x_k) \begin{smallmatrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{smallmatrix} \cdots$$

in Set. Of course, both diagrams have the same limit in Set, but that is not the case any more if each $F(x)$ were a category instead of a set. For simplicity, assume that the covering now consists of a single map $f : x \longrightarrow y$. This is not as big a restriction as it seems when the category $\mathcal{C}$ is extensive, but in any case such unary coverings are all the generality needed for Paper II. For a pseudo-functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ the sheaf condition we considered above becomes a *stack* condition, stating that

$$F(y) \; -f^* \rightarrow \; F(x) \; \begin{smallmatrix} - s^* \rightarrow \\ \leftarrow i - \\ - t^* \rightarrow \end{smallmatrix} \; F(x \times_y x) \; \begin{smallmatrix} \leftarrow p^* - \\ \leftarrow m - \\ \leftarrow q^* - \end{smallmatrix} \; F(x \times_y x \times_y x)$$

be a 2-categorical equaliser in the appropriate sense, called the *descent object* $\operatorname{Desc}(F,f)$ of the truncated simplicial diagram displayed above. Concretely, an object of the category $\operatorname{Desc}(F,f)$ consists of an element a of $F(x)$ together with *descent data*: a morphism $\alpha : s^*a \longrightarrow t^*a$ in $A_{x\times_y x}$ which satisfies the unit and cocycle conditions

$$i^*\alpha \cong 1_a \qquad\qquad (q^*\alpha)\circ(p^*\alpha) \cong m^*\alpha$$

in A_x and $A_{x\times_y x\times_y x}$ respectively. The intuition here is that a lives over x and that the data of α is what is need for it to potentially 'descend along f' and live over y. Certainly, each element of $F(y)$ comes with canonical descent data and therefore there is a functor $F(y) \longrightarrow \operatorname{Desc}(F,f)$. This is simply the factorisation of the cone on $F(y)$ through the universal cocone.

$$\begin{array}{ccccccc}
F(y) & \longrightarrow & F(x) & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & F(x\times_y x) & \begin{array}{c}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{array} & F(x\times_y x\times_y x)\\
\downarrow & \nearrow & & & & & \\
\operatorname{Desc}(F,f) & & & & & &
\end{array}$$

When the map $F(y) \longrightarrow \operatorname{Desc}(F,f)$ is an equivalence we say that F satisfies descent for f, that f is of descent type for F, or that F is a stack for the covering $(f : x \longrightarrow y)$.

2.3 Summary of Paper II

The relationship between locales and toposes lies at the heart of categorical logic and topos theory. Locales, which are the spatial embodiments of propositional geometric theories, embed 2-fully faithfully in toposes. In Paper II we prove a sort of converse: toposes embed 2-fully faithfully in (pre)stacks over locales. At a logical level, this means that the data of a geometric theory is equivalent to that of all its propositional extensions, or even to all its models in sheaf toposes. Specifically, we show that the nerve of $\mathrm{Sh} : \mathsf{Loc} \longrightarrow \mathsf{Topos}$ is fully faithful in the 2-categorical sense.

Theorem. For all toposes $\mathcal{E}$ and $\mathcal{F}$, the functor

$$\mathsf{Topos}(\mathcal{F},\mathcal{E}) \longrightarrow \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathsf{Topos}(\mathrm{Sh}(-),\mathcal{F}), \mathsf{Topos}(\mathrm{Sh}(-),\mathcal{E}))$$

is an equivalence.

This can also be read as a density theorem: every topos $\mathcal{E}$ is completely determined, up to equivalence, by geometric morphisms $\mathrm{Sh}(X) \to \mathcal{E}$ for varying locales X, the *localic points* of $\mathcal{E}$.

Paper II is an additional instalment in the broader program of representing toposes by localic or topological data (that Paper I also fits). Nevertheless, it does offer a change of perspective on that program in that it moves away from the more compact non-canonical usual groupoidal presentations and instead encodes a topos by *all* its localic data, thereby guaranteeing the intrinsicality of the representing objects at the cost of increasing their size.

The proof relies on two main ingredients. The first is the general 2-categorical theory of sites and stacks, building on the foundational work of Street [Str82b; Str82a] and Bunge–Paré [BP79]. More specifically, we need a key technical result concerning the categorification of dense monomorphisms. Recall that if J is a Grothendieck topology on a category $\mathcal{C}$, a monomorphism $A \hookrightarrow B$ between presheaves on $\mathcal{C}$ is said to be *J-dense* if its image $\mathsf{a}(A) \hookrightarrow \mathsf{a}(B)$ under the sheafification functor is an isomorphism. The analogous notion for stacks is that of a J-dense fully faithful morphism between prestacks. In Paper II, we characterise such morphisms as those whose pseudo-pullbacks along representable prestacks are J-covering sieves. This is the most technical proof of the paper given the amount of coherences inherent when one deals with pseudo-functors, pseudo-natural transformations, and modifications.

The second ingredient is the profound theorem of Joyal and Tierney [JT84], which shows that open surjections of toposes are effective descent morphisms. If one sees the 2-category of locales as a 2-site with the *open topology*, whose coverings are open surjections, then this theorem can be reframed as showing that each topos induces a stack on locales by way of its categories of localic points. The ideas in the proof strategy follow arguments of Bunge [Bun90], where analogous results were obtained in the (2,1)-core of the 2-categories we consider, but are now 'laxified' and applied in a fully (2,2)-setting.

Having shown that $\mathsf{Topos} \hookrightarrow \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}$ is a 2-fully faithful embedding, we turn to the question of whether it admits a left adjoint. This relates to the work of Di Liberti [Di 22], who constructs such adjoint in the similar situation of modest prestacks over topological spaces. We discuss this at the end of the paper, showing that while $\mathsf{Topos}(\mathrm{Sh}(-), \mathcal{E})$ is not in general a small prestack, which bars us from applying the result of loc. cit., but it is always a small stack.

3 Toposes as ultracategories

As mentioned in the summary of Paper II, one cannot hope to recover all toposes from their categories of points. The reason is twofold: some toposes simply do not have points at all, and even for toposes with enough points the 1-category of points gives very scant information about the topos. Think for example of the topos of sheaves on a topological space X, for which the category of points is simply the set of points of (the soberification of) X equipped with the specialisation order, so if X is Hausdorff, say, the category of points of $\mathrm{Sh}(X)$ is only the underlying set of X. We need more information to recover the topology of X. This is where the works of [Bar70] (for topological spaces) and [Mak87] (for coherent toposes) come in.

3.1 Ultraconvergence

It has been noticed by Barr [Bar70] that topological spaces admit an algebraic description as 'relational beta-modules'. Later, this result was recast in the 'generalised multicategorical point of view' by Cruttwell–Shulman [CS10]. Essentially: a topological space is given by a set of points and (multi)arrows between them encoding the (ultra)convergence of these points.

Recall that an ultrafilter σ on a set S is a lattice homomorphism $\sigma : 2^S \longrightarrow 2$, which gives rise to a notion of σ-large subsets of S (those mapped to 1). We can think of an ultrafilter σ as a 'convergence direction' on the set S: a subset of S is large, if it contains a potential point at infinity in the direction of σ. A σ-family (or ultrafamily) of points $(a_s)_{s:\sigma}$ of a set A is an equivalence class of families $(a_s)_{s\in S}$ of A where two such families are identified if they coincide on a σ-large subset of S. One can think of such a $(a_s)_{s:\sigma}$ as a sequence of points directed along σ.

In a topological space X, we can define the *ultraconvergence* relation

$$a \preccurlyeq_\sigma b_s$$

with σ an ultrafilter on a set S, a a point of X, and $(b_s)_{s:\sigma}$ a σ-family of points of X. The σ-family $(b_s)_{s:\sigma}$ is said to converge to a if each open neighbourhood of a contains a σ-large amount of the (b_s), or equivalently that a belongs to the closure of each σ-large collection of the (b_s).

It turns out that this notion of convergence is general enough to recover the topology of the space. More precisely, the data of a topology on a set of points can be completely reframed as such a relation satisfying some axioms. As the notation $a \preccurlyeq_\sigma b_s$ suggests, when $(b_s)_{s:\sigma}$ converges to a we think of it as there being a multi-arrow from a to $(b_s)_{s:\sigma}$. Abstracting this situation, we end up with *vu-posets*. A vu-poset is given by a set of points together with a relation $a \preccurlyeq_\sigma b_s$ between points and ultrafamilies, satisfying an identity axiom, $a \preccurlyeq_\star a$, and a transitivity axiom, the analogue of the composition in a category, so that $\preccurlyeq$ behaves like an order. There is also the matter of reindexings: in a topological space, the eventual limit points of an ultrafamily $(b_s)_\sigma$ depend only on its image, which needs to be reflected in vu-posets as well. As a result, we also ask that $(b_s)_{s:\sigma}$ converges to a if and only if its image $(x)_{x:b_*\sigma}$ converges to a, i.e., $a \preccurlyeq_\sigma b_s$ just if $a \preccurlyeq_{x:b_*\sigma} x$. This fact actually lets us present the convergence data on a topological space as a relation between points and ultrafilters (instead of ultrafamilies) as in the first approach of Barr.

3.2 Ultracategories

The notion of ultracategory was introduced by Makkai in [MR77]. The goal was to categorify Stone duality: the construction that takes a first-order theory to its category of models is not conservative — we lose information about the theories. However, if we take into account the ultraproduct operation that exists on the models of a first-order theory, then the resulting structure, called an ultracategory, of models is enough to recover the theory. This result, dubbed *Strong Conceptual Completeness* by Makkai, can be understood from a toposic point of view: the ultracategorical structure on the models encodes a categorification of the ultraconvergence relation explained above, which is enough to recover the classifying topos of the theory.

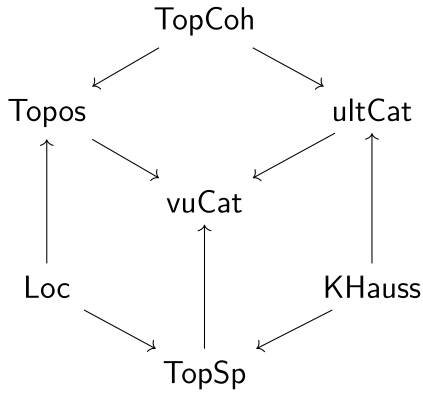
However, the above only applies for coherent logic, since models of a geometric theory are not necessarily closed under ultraproducts. Put another way, not every topos admits an ultracategory structure on its points. This is one of the main motivations behind the three independent recent works [Saa25; Ham25; GMT26], which all introduce an extension of ultracategories, based on the vu-posets described above, flexible enough to apply to the points of any topos. Of course, each author came up with a different name, and in Paper III we opt for *vu-categories* (a shortening of virtual ultracategories).

These are a categorification of the relation $\preccurlyeq$ above to the points of any topos $\mathcal{E}$. Given points p and $(q_s)_{s:\sigma}$ of $\mathcal{E}$, we let a *vu-arrow* from p to $(q_s)_{s:\sigma}$, which we denote by $p \longrightarrow\!\!\!\!\!\prec_\sigma (q_s)_{s:\sigma}$, be a natural transformation

$$p^* \Rightarrow \int_{s:\sigma} {q_s}^* \quad \text{from} \quad \mathcal{E} \xrightarrow{p^*} \mathrm{Set} \quad \text{to} \quad \mathcal{E} \xrightarrow{{q_s}^*} \mathrm{Set}^S \xrightarrow{\int_\sigma} \mathrm{Set}.$$

That is, for each sheaf A of $\mathcal{E}$, a map taking each $a \in X_p$ to an ultrafamily $(a_s \in A_{q_s})_{s:\sigma}$. Without going into all the details, the data of these vu-arrows fit into a multi-category-like object The main difference with vu-posets is that in vu-categories an ultrafamily of points can converge to another point in *more than one way*. Another thing that partially breaks down is that one cannot identify an ultrafamily with its image as easily as before.

We can summarise the situation in the following diagram: vu-categories categorify topological spaces, abstract categories of points of toposes, and virtualise the ultraproduct in ultracategories.



The original result of Makkai [Mak87], later revisited by Lurie [Lur18], showed that the functor $\mathsf{TopCoh} \longrightarrow \mathsf{ultCat}$ is 2-fully faithful. In [Saa25; Ham25; GMT26], three different proofs of a generalisation of this result have been given: the functor $\mathsf{Top}_{\mathrm{wep}} \longrightarrow \mathsf{vuCat}$, sending a topos with enough points to its vu-category of points, is 2-fully faithful.

Hence one can reconstruct a topos with enough points $\mathcal{E}$ from its vu-category of points. Concretely, from the vu-category $\mathcal{X} \coloneqq \mathrm{pt}(\mathcal{E})$ of points of $\mathcal{E}$, the data of $\mathcal{E}$ emerges as the hom-category $\mathsf{vuCat}(\mathcal{X}, \mathrm{Set})$ of vu-functors from the vu-category $\mathcal{X}$ to Set. These are naturally thought of as vu-sheaves or étale spaces over $\mathcal{X}$. Hence, any topos with enough points is equivalent to the vu-sheaves over its points.

3.3 Summary of Paper III

One of the main insights of this approach is that vu-categories, and by extension toposes with enough points, have a multi-categorical nature; one can thus transpose standard categorical notions to the vu-categorical setting. Since toposes encode categorified spaces through their categorified opens, i.e. sheaves, geometrical constructions and concepts simply have to be expressed in the language of opens and sheaves, even when the point-centric perspective simplifies the intuition and computations. The reconstruction theorem suggests that it is possible to give full characterisations of geometrical notions pertaining to toposes with enough points in terms of their vu-categories of points. In Paper III, we undertake the task of giving such characterisations for the usual classes of geometric morphisms, which we translate in the vu-categorical world.

This paper focuses on an understanding of the usual classes of geometric morphisms: embeddings, surjections, localic and hyperconnected maps, at the level of vu-categories of points. The counterparts of embeddings and localic morphisms are given by straightforwardly by fully faithful and faithful vu-functors, the treatment of hyperconnected and surjective geometric morphisms requires more care.

Let us focus only on the surjective case here. In a topos $\mathcal{E}$ with enough points, one can always find some small set of points that is separating, i.e. a surjective map $\mathrm{Set}^A \twoheadrightarrow \mathcal{E}$. But most often $\mathcal{E}$ actually has a proper class of points and $A \longrightarrow \mathrm{pt}(\mathcal{E})$ is simply not essentially surjective on points, so we need a more general notion of surjection between vu-categories. It is worth pointing out that, logically speaking, such a surjection $\mathrm{Set}^A \twoheadrightarrow \mathcal{E}$ amounts to a choice of a conservative family of models of $\mathcal{E}$. Hence, we set out to understand when such a family of models is conservative for the theory at hand, from a vu-categorical point of view.

We slightly extend the scope of this question: given $A \subseteq \mathrm{pt}(\mathcal{E})$ a collection of points of $\mathcal{E}$, can we characterise vu-categorically the points that belong to the subtopos $\mathcal{E}_{|A} \subseteq \mathcal{E}$, or equivalently characterise the points of $\mathcal{E}$ that belongs to each subtopos of $\mathcal{E}$ containing the points of A. From a logical point of view, we aim to characterise the models of the geometric theory of $\mathcal{E}$ satisfying the same geometric sequents as the models in A, i.e., satisfying the theory of A.

In the paper, we give a full answer to those questions in terms of a vu-categorification of the usual notions of retraction. This extends the known fact that, for $\mathcal{E} = \mathrm{Set}^{\mathcal{C}}$ a presheaf topos, a collection A of objects of $\mathcal{C}$ separates $\mathrm{Set}^{\mathcal{C}}$ if and only if every object of $\mathcal{C}$ is a retract of some object in A. Essentially this is because retractions are absolute colimits and as a result a presheaf on $\mathcal{C}$ is determined by its values on A. This gives only a partial answer for $A \subseteq \mathrm{ob}(\mathcal{C})$ but not for arbitrary collections in $\mathrm{Ind}(\mathcal{C}) = \mathrm{pt}(\mathrm{Set}^{\mathcal{C}})$.

So far we have taken the categorical point on view on vu-categories, but we must not forget that they are also topological objects. If X is a topological space, and $A \subseteq X$ a subset of points, and that we want to know when the inclusion $A \hookrightarrow X$ induces an equivalence at the level of sheaf toposes, we can turn to the work of Johnstone [Joh80] in the notion of *subclosure*. The subclosure of A consists of the points p of X such that any locally closed set (an intersection of a closed set with an open set) containing p intersects A, equivalently such that any open neighbourhood of p meets the closure of p, or equivalently again such that $p \in \mathrm{cl}(A \cap \mathrm{cl}(\{p\}))$. The notion of vu-retractions arises naturally from this last observation. It can be rewritten as $p \preccurlyeq_\sigma (a_s)$ and $(a_s \preccurlyeq p)$ for an ultrafamily (a_s) of points in A.

In the paper, we introduce a notion of *vu-retractions* extending at the same time the notions of retractions in a category and subclosure in a topological space, and we show that given a class of points $A \subseteq \mathrm{pt}(\mathcal{E})$ of a topos, a point of $\mathcal{E}$ lies in the subtopos $\mathcal{E}_{|A}$ if and only if it is a vu-retract of points in A.

Bibliography

[AF13] Steve Awodey and Henrik Forssell. First-order logical duality. *Annals of Pure and Applied Logic* 164(3) (2013). DOI: 10.1016/j.apal.2012.10.016.

[Bar70] Michael Barr. 'Relational algebras'. *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 1970.

[BJ98] Carsten Butz and Peter T. Johnstone. Classifying toposes for first-order theories. *Ann. Pure Appl. Logic* 91(1) (1998). DOI: 10.1016/S0168-0072(97)00042-0.

[BM98] Carsten Butz and Ieke Moerdijk. Representing topoi by topological groupoids. English. *J. Pure Appl. Algebra* 130(3) (1998). DOI: 10.1016/S0022-4049(97)00107-2.

[BP79] Marta Bunge and Robert Paré. Stacks and equivalence of indexed categories. *Cah. Topologie Géom. Différ. Catégoriques* 20 (1979). URL: https://eudml.org/doc/91222.

[Bun90] Marta Bunge. An application of descent to a classification theorem for toposes. English. *Math. Proc. Camb. Philos. Soc.* 107(1) (1990). DOI: 10.1017/S0305004100068365.

[Car18] Olivia Caramello. *Theories, sites, toposes. Relating and studying mathematical theories through topos-theoretic 'bridges'*. Oxford: Oxford University Press, 2018. DOI: 10.1093/oso/9780198758914.001.0001.

[CS10] G. S. H. Cruttwell and Michael A. Shulman. A unified framework for generalized multicategories. English. *Theory Appl. Categ.* 24 (2010). URL: https://eudml.org/doc/228027.

[Di 22] Ivan Di Liberti. The geometry of Coherent topoi and Ultrastructures. 2022. URL: https://arxiv.org/abs/2211.03104.

[DLY26] Ivan Di Liberti, Peter LeFanu Lumsdaine and Errol Yuksel. Toposes vs Localic Groupoids: a unified treatment of covering theorems. *Preprint*. 2026.

[GMT26] Sam van Gool, Jérémie Marquès and Umberto Tarantino. Toposes with enough points as categories of étale spaces. 2026. URL: https://arxiv.org/abs/2508.09604.

[Hak72] Monique Hakim. *Topos anneles et schemas rélatifs*. French. Vol. 64. Ergeb. Math. Grenzgeb. Springer-Verlag, Berlin, 1972.

[Ham25] Ali Hamad. Generalised ultracategories and conceptual completeness of geometric logic. 2025. URL: https://arxiv.org/abs/2507.07922.

[Joh02] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium*. Vol. 43,44. Oxford Logic Guides. New York: The Clarendon Press Oxford University Press, 2002.

[Joh80] Peter T. Johnstone. Open maps of toposes. *Manuscr. Math.* 31 (1980). DOI: 10.1007/BF01303275. URL: https://eudml.org/doc/154704.

[Joh83] Peter T. Johnstone. The point of pointless topology. *Bull. Am. Math. Soc., New Ser.* 8 (1983). DOI: 10.1090/S0273-0979-1983-15080-2.

[Joh85] Peter T. Johnstone. 'How general is a generalized space?' *Aspects of topology*. Vol. 93. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1985. DOI: 10.1017/CBO9781107359925.004.

[JT84] André Joyal and Myles Tierney. An extension of the Galois theory of Grothendieck. *Mem. Amer. Math. Soc.* 51(309) (1984). DOI: 10.1090/memo/0309.

[Lur18] Jacob Lurie. Ultracategories. 2018. URL: https://www.math.ias.edu/~lurie/papers/Conceptual.pdf.

[Mak87] M. Makkai. Stone duality for first order logic. *Advances in Mathematics* 65(2) (1987). DOI: https://doi.org/10.1016/0001-8708(87)90020-X. URL: https://www.sciencedirect.com/science/article/pii/000187088790020X.

[Moe88] Ieke Moerdijk. The Classifying Topos of a Continuous Groupoid. I. *Transactions of the American Mathematical Society* 310(2) (1988). DOI: 10.2307/2000984.

[MR77] Michael Makkai and Gonzalo E. Reyes. *First order categorical logic. Model-theoretical methods in the theory of topoi and related categories*. Vol. 611. Lect. Notes Math. Springer, Cham, 1977. DOI: 10.1007/bfb0066201.

[Saa25] Gabriel Saadia. Extending conceptual completeness via virtual ultracategories. 2025. URL: https://arxiv.org/abs/2506.23935.

[SGA4.1] Michael Artin et al., eds. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Tome 1: Théorie des topos.* French. Vol. 269. Lect. Notes Math. Springer, Cham, 1972. DOI: 10.1007/BFb0081551. URL: http://fabrice.orgogozo.perso.math.cnrs.fr/SGA4/index.html.

[Str82a] Ross Street. Characterization of bicategories of stacks. *Category theory. Applications to algebra, logic and topology, Proc. int. Conf., Gummersbach 1981, Lect. Notes Math. 962, 282-291 (1982).* 1982. DOI: 10.1007/bfb0066909.

[Str82b] Ross Street. Two-dimensional sheaf theory. *Journal of Pure and Applied Algebra* 23(3) (1982). DOI: https://doi.org/10.1016/0022-4049(82)90101-3. URL: https://www.sciencedirect.com/science/article/pii/0022404982901013.

[SY26] Gabriel Saadia and Errol Yuksel. Geometric morphisms of virtual ultracategories. *Preprint.* 2026.

[Wri23] Joshua L. Wrigley. On topological groupoids that represent theories. *Preprint, arXiv:2306.16331 [math.CT] (2023)*. 2023. URL: `https://arxiv.org/abs/2306.16331`.

[Wri25] Joshua L. Wrigley. Topoi with enough points and topological groupoids. *J. Pure Appl. Algebra* 229(10) (2025). Id/No 108073. DOI: `10.1016/j.jpaa.2025.108073`.

[Yuk26] Errol Yuksel. Toposes as Localic Stacks. *Preprint*. 2026.